3-Factors of 3-Factorization of $K_{3,3,3,\cdots,3}$ with n-Partite Sets for All Even Integers $n \geq 2$

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***Abstract:-*A factorization of a graph G is a set of spanning sub-graph of G that are pairwise edge-disjoint and whose union is G. Factorization is one of the most active research areas in Graph Theory. In our previous work, 2-factors of 2-factorization of $K_{2,2,2,\cdots,2}$ and $K_{2^r,2^r,2^r,\cdots,2^r}$ has been constructed by using degree factors. In this work, by considering degree factorization, a theorem has been proved to obtain 3-factors of 3-factorization of the complete multipartite graphs $K_{3,3,3,\cdots,3}$.**

***Keywords:* Complete n-partite graph, Factors, Factorization.**

I. INTRODUCTION

Graph Theory is an important area in mathematics with many applications. A graph is a structure with vertices and edges. A simple graph G consists of a non-empty finite set$V(G)$of elements called vertices (or nodes) and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $v(G)$ called edges [7]. The concept of factorization was introduced by Kirkman in 1847. In graph theory, a factor of a graph G is a spanning sub-graph of G which is not totally disconnected. A graph factorization of G is defined as a partition of edges of G into disjoint factors. Akiyama and Kano introduced two different types of factors which are called *degree* factors and *component* factors. A factor F described in terms of its degrees will be called a degree factor. On the other hand, if the factor is described using some other graphical method, it is called a component factor. The degree factors have been used for our research (A factor described in terms of its degrees).[1,4]

Application of graph factorization are involved in travelling salesman problem, coding theory, time scheduling in Multi-hop Networks, cooperative localization, Round-Robing tournaments etc.

Definition 1

A graph $K_{3,3,3,\cdots,3}$ is*n-partite* if its vertices can be partitioned into n-sets (each with 3 points) in such a way that no edge joins vertices in the same set.

Definition 2

A *complete n-partite graph* is a simple n-partite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite sets.[7]

Definition 3.[1, p 59; 2, p 403].

A factor that is n-regular (degree of each vertex is n) is called an *n-factor(or k-regular factors).*

Definition 4.[3, p 403].

If a graph G can be represented as the edge-disjoint union of factors $F_1, F_2, \cdots, F_n$, then $\{F_1, F_2, \cdots, F_n\}$ is refered as a *factorization* of a graph G.

Definition 5

A tournament schedule is an arrangement of all pairs of teams into the minimum number of rounds so that every team plays against every other team once. Suppose that there are n teams in a competition. Then, the number of matches to be played is

$$^{n}C_2 = \frac{n(n-1)}{2}. \qquad [7]$$

II. MATERIALS AND METHODS

Theorem: Acomplete multipartite graph with $2n$ partite sets of the form $K_{3,3,3,\cdots,3}$ has $\frac{^{2n}C_2}{n} = 2n-1$, 3-factors for a 3-factorization for $2n$ partite sets.

Proof. Using Mathematical induction,

When $n = 1$; 2 partite sets of the form $K_{3,3}$ has $\frac{^{2}C_2}{1} = (2-1) = 1$; 3-factors of a3-factorization.

When $n = 2$; 4 partite sets of the form $K_{3,3,3,3}$ has $\frac{^{4}C_2}{2} = (4-1) = 3$; 3-factors of a 3-factorization.

Assume that the result is true for $n = k$; $K_{3,3,3,\ldots,3}$

i.e. $2k$-partite sets of the form $K_{3,3,3,\ldots,3}$ have $\frac{^{2k}C_2}{k} = (2k-1)$; 3-factors of a3-factorization.

For$n = k + 1$,

In each case, two newpartite sets are added to the previous multipartite graph. If one, $2k$-partite set is fixed and is joined with the added two newpartite sets, then there are ${}^{2}C_1$ ways that they can be joined.

Therefore, $2(k+1)$partite sets of the form $K_{3,3,3,\ldots,3}$ have $(2k-1)+{}^{2}C_1=(2k+1)$; 3-factors of a3-factorization.

Hence, the result is true for $n=k+1$. Hence, by mathematical induction the result is true for all $n \in \mathbb{N}$.

III. RESULTS AND DISCUSSION

Tournament scheduling technique has been used for this construction as follows:

Suppose there are n teams in a league, in a season, every team plays against every other team exactly once, then number of matches played is ${}^{n}C_2=\frac{n(n-1)}{2}$. If n is even, $\frac{n}{2}$ in each round and $(n-1)$ rounds are required.

Considering n partite sets as $a_1, a_2, \cdots, a_n$, using the above technique, $\frac{n}{2}$ pairs$a_i a_{i+1}; i=1,2,\ldots,(n-1)$, can be constructed and those can be considered as one 3-factor of a3-factorization. According to this construction a_i can be paired with a_j where, $j=1,2,\ldots,n$except i. i.e. $(n-1)$ different ways. Hence, $(n-1)$ different 3-factors can be obtained. Since $(n-1)$ rounds are required, $(n-1)$ number of 3-factors of 3-factorization can be obtained.

This has been illustrated by the following example, when $n=6$, a 3-factor of $K_{3,3,3,3,3,3}$ is given below.

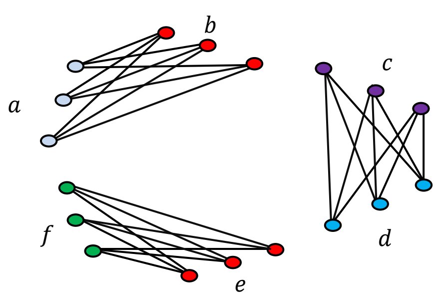

Figure 1: A 3-factor of $K_{3,3,3,3,3,3}$ graph.

Labeling 6 partite sets as a, b, c, d, e, f,one 3-factor of 3-factorization can be considered asab, cd, efwhich corresponds to matches in one round. Paring a with b, c, d, e and$f$; and using tournament scheduling technique five, 3-factors of 3-factorization can be obtained.

IV. CONCLUSIONS

Most of the research work on factorization is on complete graphs and complete bipartite graphs. In our research,3-factors of 3-factorizations for different values of n have been constructed using complete multipartite graphs $K_{3,3,3,\cdots,3}$ with n partite sets. Furthermore, tournament scheduling technique can be used to obtain the number of 3-factors of 3-factorization.As future work, try to generalize this work for $K_{3^r,3^r,3^r,\cdots,3^r}$ with n partite sets for positive integers $r, n \geq 2$.

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