A Study of W_6-Curvature Tensor in Sasakian Manifold

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Abstract: - **Pokhariyal have introduced W_6 curvature tensors to study their properties. In this paper properties of W_6-curvature tensor are studied in Sasakian manifold. The following geometric properties are being studied; flatness, semi-symmetric, symmetric and recurrence on Sasakian manifold. The result show that W_6-semi symmetry and symmetric are W_6-flat manifold while W_6-recurrent manifold (under some condition) is symmetric and semi symmetric manifold.**

I. INTRODUCTION

Let (M,F,T,A,g) be (2n+1)-dimensional almost contact metric manifold consisting of a (1, 1) tensor field F,a vector field T,a 1-form A and a Riemannian metric g which satisfies:

(1.1) A(T)=1

(1.2) $\bar{X}$ +X=A(X)T where $\bar{X}$ =F(X) and $\bar{T}$ =F(T) then A($\bar{X}$)=0 and $\bar{T}$=0 (Pokhariyal (1998))

(1.3) g($\bar{X},\bar{Y}$)=g(X,Y) – A(X)A(Y)

(1.4) g(X,T)= A(X) and g(X,$\bar{Y}$)= - g($\bar{X}$,Y)

where X and Y are arbitrary vector fields on M.

An almost contact metric manifold is contact metric manifold if

(1.5) dA(X,Y)=g(X,$\bar{Y}$) and almost contact metric manifold is K-contact metric manifold if

(1.6) ∇_xT= - $\bar{X}$

where∇ is levi-civita connection

An almost contact metric manifold isK-contact metric manifold in a sasakian manifold if

(1.7) (∇_xF)Y=g(X,Y)T – A(Y)X

A sasakian manifold is a K-contact but the converse is only true if dimension is 3.

A contact metric manifold is sasakian if and only if

(1.8) R(X,Y)T=A(Y)X – A(X)Y

In sasakian manifold (M,F,T,A,g) we easily get

(1.9) R(T,X)Y= g(X,Y)T – A(Y)X

In generally in (2n+1)- dimensional sasakian manifold with the structure (F,T,A,g) we have

(1.10) rank(F) =n-1

(1.11) R'(X,Y,Z,U)= g(R(X,Y),Z,U)

=g({g(Y,Z)X – g(X,Z)Y},U)

=g(Y,Z)g(X,U) – g(X.Z)g(Y,U)

=g(Y,Z)A(X) – g(X,Z)A(Y)

where R is Riemannian curvature tensor.

(1.12) S(X,Y) = g (QX,Y) = (n-1)g(X,Y)= Ric (X,Y)

Where Q is Ricci operator and Ric (X,Y) denote Ricci tensor

II. W_6-CURVATURE TENSOR IN SASAKIAN MANIFOLD

(2.1) W_6'(X,Y,Z,U) = R'(X,Y,Z,U) + $\frac{1}{n-1}$ [g(X,Y)Ric (Z,U) – g(X,U)Ric(Y,Z)]

Definition 2.1

A sasakian manifold M is said to be flat if the Riemannian curvature tensor vanishes identically i.e R(X,Y)Z= 0

Definition 2.2

A sasakian manifold M is said to be W_6-flat if W_6 curvature tensor vanishes identically i.e W_6(X,Y)Z=0

Theorem 2.3

A W_6-flat sasakian manifold is a flat manifold.

Proof

If our hypothesis is true then W_6=0 in

W_6(X,Y)Z= R(X,Y)Z + $\frac{1}{n-1}$ [g(X,Z)Y- S(Y,Z)X]

Or

W_6'(X,Y,Z,U) = R'(X,Y,Z,U) + $\frac{1}{n-1}$ [g(X,Y)Ric (Z,U) – g(X,U)Ric(Y,Z)]

Therefore, if sasakian manifold M is W_6-flat then

(2.2) 0 = R'(X,Y,Z,U) + $\frac{1}{n-1}$ [g(X,Y)Ric (Z,U) – g(X,U)Ric(Y,Z)]

Or

$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)Ric(Y,Z) - g(X,Y)Ric(Z,U)]$

Where in (1.12) we have $(n-1)g(X,Y)= Ric(X,Y)$

Using (1.12) in equation (2.2), we get

$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,U)(n-1)g(Y,Z) - g(X,Y)(n-1)g(Z,U)]$

$= \frac{n-1}{n-1}[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)]$

Or

(2.3) $R'(X,Y,Z,U) = [g(X,U)g(Y,Z) - g(X,Y)g(Z,U)]$

But in sasakian manifold we have

$R'(X,Y,Z,U) = [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$

Thus, for this to hold, we must have

$R'(X,Y,Z,U) = 0$ since

$[g(X,U)g(Y,Z) - g(X,Y)g(Z,U)] \neq [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$

hence the theorem proved

III. W_6-SEMI-SYMMETRIC SASAKIAN MANIFOLD

De and Guha(1992) gave definition of semi symmetric as

$R(X,Y)R(Z,U)V=0$

Definition 3.1

A sasakian manifold M is said to be W_6-semi symmetric if $R(X,Y)W_6(Z,U)V=0$ (3.1)

Theorem 3.2

A W_6- semi symmetric sasakian manifold is a W_6- flat manifold.

Proof

If the sasakian space is W_6- semi symmetric then $R(X,Y)W_6(Z,U)V=0$

$R(X,Y)W_6(Z,U)V= g(Y, W_6(Z,U)V)X – g(X, W_6(Z,U)V)Y = 0$

- ⇨ $g(Y, W_6(Z,U)V)X – g(X, W_6(Z,U)V)Y = 0$
- ⇨ $W_6'(Y,Z,U,V)X - W_6'(X,Z,U,V)Y = 0$
- ⇨ $g(W_6'(Y,Z,U,V)X,T) – g(W_6'(X,Z,U,V)Y,T) = 0$
- ⇨ $W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) = 0$ (3.2)

But since $A(X)\neq 0$ and $A(Y)\neq 0$ then it follows that

$W_6'(Y,Z,U,V)=0$ and $W_6'(X,Z,U,V) = 0$ hence the theorem proved.

IV. W_6-SYMMETRIC SASAKIAN MANIFOLD.

Chaki and Gupta (1963) gave the definition of a conformally symmetric manifold as

$\nabla_u C=0$ where C is conformal curvature tensor

Definition 4.1

A sasakian manifold M is said to be W_6-symmetric if $\nabla_u W_6(X,Y)Z= W_6'(U,X,Y)Z =0$ (4.1)

Theorem 4.2

A W_6- symmetric and W_6- semi symmetric sasakian manifold is a flat manifold.

Proof

If the sasakian space is a W_6- symmetric then it follows

$\nabla_u W_6(X,Y)Z= R(X,Y)W_6(Z,U)V-W_6(R(X,Y)Z,U)V-W_6(Z,R(X,Y)U)V -W_6(Z,U)R(X,Y)V =0$ (4.2)

Computing each of above four term and subject them to same conditions we have ;

$R(X,Y)W_6(Z,U)V = g(Y, W_6(Z,U)V)X - g(X, W_6(Z,U)V)Y$

$= W_6'(Y,Z,U,V)X - W_6'(X,Z,U,V)Y$

$g(R(X,Y), W_6(Z,U)V, T) = g(W_6'(Y,Z,U,V)X,T) – g(W_6'(X,Z,U,V)Y,T)$

$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y)$ (4.3)

Again if

$W_6(X,Y)Z= R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y- S(Y,Z)X]$

Now

$W_6(R(X,Y)Z,U)V= R(R(X,Y)Z,U)V + \frac{1}{n-1}[g(R(X,Y)Z ,V)U - S(U,V)R(X,Y)Z]$

But

$S(U,V)= (n-1)g(U,V)$ and $R'(X,Y,Z,U) = g(R(X,Y)Z,U)$

SO

$W_6(R(X,Y)Z,U)V= R(R(X,Y)Z,U)V + \frac{1}{n-1}[R'(X,Y,Z,V)U - (n-1)g(U,V)R(X,Y)Z]$

$= g(U,V)R(X,Y)Z – g(R(X,Y)Z,V)U + \frac{1}{n-1}R'(X,Y,Z,V)U - g(U,V)R(X,Y)Z$

$= \frac{1}{n-1}R'(X,Y,Z,V)U – g(R(X,Y)Z,V)U$

$=\frac{1}{n-1}R'(X,Y,Z,V)U - R'(X,Y,Z,V)U$ (4.4)

Also

$W_6(Z,R(X,Y)U)V= R(Z,R(X,Y)U)V + \frac{1}{n-1}$ [g(Z ,V) R(X,Y)U - S(R(X,Y)U,V)Z]

$= g(R(X,Y)U,V)Z – g(Z,V)R(X,Y)U + \frac{1}{n-1}$ [g(Z ,V)R(X,Y)U-(n-1)g(R(X,Y)U,V)Z]

$=\frac{1}{n-1}$ g(Z ,V)R(X,Y)U - g(Z,V)R(X,Y)U (4.5)

Also

$W_6(Z,U)$ R(X,Y)V = R(Z,U) R(X,Y)V + $\frac{1}{n-1}$ [g(Z, R(X,Y)V)U- S(U, R(X,Y)V)Z]

$= g(U,R(X,Y)V)Z – g(Z,R(X,Y)V)U + \frac{1}{n-1}$ g(Z, R(X,Y)V)U- g(U, R(X,Y)V)Z]

$= \frac{1}{n-1}$g(Z, R(X,Y)V)U - g(Z,R(X,Y)V)U (4.6)

Next in (4.2) we put (4.3), (4.4), (4.5) and (4.6) and we have

$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) – \{\frac{1}{n-1}$ R'(X,Y,Z,V) U - R'(X,Y,Z,V)U+$\frac{1}{n-1}$ g(Z ,V)R(X,Y)U - g(Z,V)R(X,Y)U + $\frac{1}{n-1}$ g(Z ,V)R(X,Y)U - g(Z,V)R(X,Y)U}

$= W_6'(Y,Z,U,V)A(X) - W_6'(X,Z,U,V)A(Y) - \frac{2-n}{n-1}$ [R'(X,Y,Z,V) U + g(Z ,V)R(X,Y)U + g(Z ,U)R(X,Y)V] =0

But since $\nabla_x W_6'(Y,Z,U,V) = 0$ and g(Z ,U) ≠ g(Z ,V) ≠ 0 => R'(X,Y,Z,V)=0

Thus follows the theorem.

Corollary 4.5.3

A W_6- symmetric sasakian manifold is always W_6- semi symmetric sasakian manifold

That is for$\nabla_X W_6'(Y,Z,U,V) =0$

⇨ $R(X,Y)W_6(Z,U)V =0$

V. W_6-RECURRENT SASAKIAN MANIFOLD

We study some geometrical properties of W_6-curvature tensor on W_6 recurrent sasakian manifold M.

$\nabla_U W_6(X,Y)Z = B(U) W_6(X,Y)Z$ (5.1)

Where B is non-zero 1-form.

If we consider a sasakian manifold M which is W_6 then we have Pokhariyal (1988)

$\nabla_U W_6(X,Y)Z =B(U) W_6(X,Y)Z$ (5.2)

Then we know that

$W_6(X,Y)Z= R(X,Y)Z + \frac{1}{n-1}$ [g(X,Z)Y- S(Y,Z)X]

$= g(Y,Z)X – g(X,Z)Y + \frac{1}{n-1}$ [g(X,Z)Y- (n-1) g(Y,Z)X]

$=g(X,Z)Y [\frac{1}{n-1} -1]$

$=\frac{2-n}{n-1}$ g(X,Z)Y (5.3)

$g(W_6(X,Y)Z,T) = g(\frac{2-n}{n-1}$ g(X,Z)Y,T)

$W'_6(X,Y,Z,T) = \frac{2-n}{n-1}$ g(X,Z)A(Y)

Note $W'_6(X,Y,Z,T) = 0$ if and only if g(X,Z)A(Y)=0.

Theorem 5.1

A W_6 recurrent sasakian manifold $R(X,Y)W_6(Z,U)V =0$ and 0 $= \frac{2-n}{n-1}$ g(X,Z)A(Y) is symmetric and semi symmetric space.

Proof

From (5.1) we have $\nabla_U W_6(X,Y)Z =B(U) W_6(X,Y)Z$

$\nabla_X W_6(Z,U)V = R(X,Y)W_6(Z,U)V - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V$ (5.4)

But we are given

$R(X,Y)W_6(Z,U)V = 0$

Now in (5.4) we have

$\nabla_X W_6(Z,U)V= - W_6(R(X,Y)Z,U)V - W_6(Z,R(X,Y)U)V - W_6(Z,U)R(X,Y)V$ (5.5)

Recall in (5.3)

$W_6(X,Y)Z =\frac{2-n}{n-1}$ g(X,Z)Y

Expanding (5.5)

$W_6(R(X,Y)Z,U)V =\frac{2-n}{n-1}$ g(R(X,Y)Z,V)U

$=\frac{2-n}{n-1}$R'(X,Y,Z,V) U (5.6)

$W_6(Z,R(X,Y)U)V =\frac{2-n}{n-1}$ g(Z,V)R(X,Y)U (5.7)

And

$W_6(Z,U)R(X,Y)V =\frac{2-n}{n-1}$ g(Z,R(X,Y)V)U

$=\frac{2-n}{n-1}$ R'(X,Y,V,Z)U (5.8)

Conbining (5.6),(5.7) and (5.8) in (5.5) we have

$\nabla_X W_6(Z,U)V= - \frac{2-n}{n-1}$R'(X,Y,Z,V) U $- \frac{2-n}{n-1}$ g(Z,V)R(X,Y)U - $\frac{2-n}{n-1}$ R'(X,Y,V,Z)U

$= \frac{n-2}{n-1}$ [R'(X,Y,Z,V)U + g(Z,V)R(X,Y)U + R'(X,Y,V,Z)U] (5.9)

Terms in R' cancels out since R'(X,Y,Z,V)= - R'(X,Y,V,Z) hence we have

$\nabla_X W_6(Z,U)V = \frac{n-2}{n-1}$ [g(Z,V)R(X,Y)U] (5.10)

Expanding (5.10) gives

$\nabla_X W_6(Z,U)V = B(X)\, W_6(Z,U)V$

$= \frac{n-2}{n-1} g(Z,V)[g(Y,U)X - g(X,U)Y]$ (5.11)

Taking inner product of (5.11) with respect to T both sides yields

$g[\nabla_X W_6(Z,U)V,T] = g[B(X)\, W_6(Z,U)V,T]$

$= \frac{n-2}{n-1} g(Z,V)[g(Y,U)g(X,T) - g(X,U)g(Y,T)]$

$= \frac{n-2}{n-1} g(Z,V)[g(Y,U)A(X) - g(X,U)A(Y)]$ (5.12)

The coefficient g(Z,V) from the initial given condition is equal to zero hence

$\nabla_X W_6(Z,U)V = B(X)\, W_6(Z,U)V = 0$ (5.13)

This completes the theorem.

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