Loss and Delay Multi Server State Dependent Queue with Discouragement, Additional Server and No-Passing

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***Abstract: -* This study deals with loss and delay multi-server no passing queueing system with discouragement. To reduce the balking behaviour of the customers in case of long queue, we incorporate additional removable servers. The customers arrive according to Poisson process and departs from the system in the same chronological order in which they join the system, due to nopassing restriction. The service facility consists of s permanent and r additional removable heterogeneous servers. The service times of the customers are exponentially distributed. The departing constraint categorizes customers in two types (i) the customers having zero service time and (ii) the customers having exponentially distributed service time. By stating the state dependent rates, the product type solution for queue size distribution at equilibrium. The analytical formulae for the expected waiting time and the difference between the expected waiting times of both type of customers are also derived. To examine the effect of variation of parameters, the sensitivity analysis is carried out. Numerical results are tabulated and displayed graphically so as to explore the advantage of additional removable servers on the performance metrics.**

***Keywords:* Multi-server Queue, No-passing, Removable additional servers, Discouragement, Queue size distribution, Expected waiting time.**

I. INTRODUCTION

The performance analysis of multi-server queues with discouragement has significant impetus on the optimal control of various service systems such as computer, manufacturing, distribution and production systems, etc. In order to discourage the balking and reneging behaviour of the customers, it is advisable to improve the service facility subject to cost constraint. The concept of nopassing in multi-server queue was introduced by Washburn (1974). Nopassing restriction implies that the customers depart from the system in the same chronological order in which they join the queue. Some other researchers have also done valuable works in this direction. Jain (1998) extended the work of infinite capacity to finite capacity and finite population models for loss and delay queueing system subject to nopassing constraint. M/M/m/K queue with nopassing and additional servers was also studied by Jain and Ghimire (1999). Finite population loss and delay queue under no passing restriction and discouragement was analysed by Sharma (2017).

The provision of additional servers is employed reduce the discouraging behaviour of the customers in case when queue size exceeds a pre-specified threshold level. Loss and delay phenomenon for nopassing queue is also of practical significance. We give here some examples such as a narrow boat lock, a remote border crossing with nopassing space, safety system, checking process at airport etc. in which the restriction of nopassing applicable. In this paper we introduce additional servers to reduce the discouragement behaviour of the customers in case of long queue for multi-server Markovian loss and delay model under the restriction of nopassing. Due to balking, the customers may not like to join the queue on seeing it very long. Also the customers may leave the system after joining it, without getting any service due to impatience, which is called the reneging behavior of the customers. Several researchers have done significant works in this direction. Makaddis and Zaki (1983) investigated $M/M/1/\infty$ queueing system with additional server. Jain (2002) considered M/M/m queue with discouragement and additional servers. Transient analysis of interdependent $M/M^{(a,d,b)}/1$ queue with discouragement and controllable arrival rate was suggested by Sharma (2017).

Loss and delay queueing phenomenon of the customers are common in many congestion problems. In such a system, some customers are lost when all servers are busy; on the other hand the remaining customers may join the queue and wait for their service when the server is busy with some other customer. Shonic (1976) developed the utility of loss and delay system for hospital administration concerning with occupancy of hospital beds. Homogeneous customers renege from invisible queues at random times under deteriorating waiting conditions, queueing systems were studied by Haviv and Ritov (2001). Single unreliable server interdependent loss and delay queueing model with controllable arrival rate under N-policy was investigated by Sharma (2013). Again, Sharma (2014) described the transient analysis of loss and delay bulk service markovian queue under n-policy. Loss and delay queue with state dependent rates under no passing restriction and discouragement was developed by Sharma (2017).

In the present chapter, we study the concept of loss and delay multi server Markovian queue with additional server and discouragement under the restriction of nopassing. In this model we have provision of some additional servers to reduce the discouraging behavior of the arriving customers. The remaining part of this chapter is organized as follows. Section 2 consists of the mathematical formulation for finite capacity and finite population models. The queue size distribution and average queue length are obtained in section 3. The expressions for the expected waiting time for both type of customers and the difference between the expected waiting times for both type of customers for finite capacity and finite population models are obtained in section 4. We also deduce some special cases for both types models in section 5. Numerical illustration is given in section 6. To validate the analytical results, sensitivity analysis has been done. Finally, the conclusion is drawn in section 7.

II. MODEL DESCRIPTION

We develop finite state dependent loss and delay multi-server queue with discouragement, nopassing and additional servers. Now we formulate the mathematical model by stating the requisite assumptions as follow: -

- There are two types of arriving customers in the system: -
  - i **Loss customer: -**The customers who depart from the system, on finding that all the servers are busy on their arrival, are called the loss customers.
  - ii **Delay customer: -**The customers who have the patience to wait for their service if all the servers are busy with other customers, are called delay customers.
- The inter-arrival times of both delay and loss type customers are exponentially distributed for with mean rate λ_1 *and* λ_2, respectively.
- There are s permanent and r additional servers to serve to the arriving customers according to FIFO service discipline. Essentially to reduce the waiting time of the customer, some additional servers are employed in the system according to pre specified rule. The number of the additional servers depends upon the number of the customers present in the system in the following manner: -
  - If there are less or equal to N customers in the system then only all permanent servers are available for service.
  - If there are greater then or equal to jN and less than (j+1)N customers in the system, then (s+j) servers provide service to the customers, where j = 1,2,3,….,r-1.
  - When there are greater than or equal to rN customers in the system then all permanent and all additional servers will provide service to the customers.
- Permanent and j^{th} (j=1,2,…,r) additional servers provide service according to exponential distribution with mean rate μ and μ_j respectively.
- From the service point of view there are also two type of customers as described below: -
  - Type A customers: - The customers who have (1-p) proportion of the total customers, have zero service time and are called type A customers.
  - Type B customers: - Remaining p proportion of the total customers require service time as exponentially distributed and are called type B customers.
- $1-\left(\frac{s+j}{n+1}\right)^{\beta}$; $j=1,2,...,r$, is the probability of balking of delay customer, so that balking of the customer depends on the number of the servers per customer. Here β denotes the degree of non-linearity.
- The customers may renege exponentially with parameter α. So that state dependent service rate when all permanent servers are busy is, given by $\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(n-\overline{s+j}\right)\alpha$ where $j=1,2,....,r$ and $jN \le n < (j+1)N$.

The following notations are used to formulate the problem mathematically:

λ_1 Arrival rate of the delay customers in the system.

λ_2 Arrival rate of the loss customers in the system.

$\lambda(n)$ State dependent arrival rate of the arriving customers when there are n units present in the system.

μ Service rate of permanent server.

μ_j Service rate of j^{th} (j=1,2,….,r) additional server.

α Reneging parameter.

β Parameter to indicate the non- linearity of balking.

P_n Steady state probability that there are n customers present in the system.

P_0 Steady state probability of the system being empty.

$E(W_A)$ Expected waiting time of type A customer in the system.

$E(W_B)$ Expected waiting time of type B customer in the system.

s Number of permanent servers in the system.

r Additional servers in the system which turn on one by one as queue size reaches to a pre-assigned threshold level.

n The number of customers in the system waiting for their service including those customers who are being served.

N Threshold value used as multiple factor to turn on the j[th] additional server when queue size crosses the level jN (j=0,1,2,…..,r).

K(M) Capacity (population) size in case of finite capacity (population) model.

D Difference between mean waiting time for type A and type B customers.

$\rho_1(\rho_2)$ Traffic intensities of the delay (loss) customers.

ρ Traffic intensity such that $\rho = \rho_1 + \rho_2$.

and $\rho_1 = \frac{\lambda_1 p}{\mu}, \rho_2 = \frac{\lambda_2 p}{\mu}$

Let ρ_1 and ρ_2 be the traffic intensities of the delay and loss customers respectively, then we have

$$\rho = \rho_1 + \rho_2 < 1 \quad and \rho = \frac{(\lambda_1 + \lambda_2)p}{\mu} < 1$$

In the finite capacity model (FCM) the state dependent arrival rates are given as follow: -

$$\lambda(n) = \begin{cases} (\lambda_1 + \lambda_2)p; & 0 \le n \le s \\ \left(\frac{s}{n+1}\right)^{\beta} \lambda_1 p; & s < n \le N \\ \left(\frac{s+j}{n+1}\right)^{\beta} \lambda_1 p; & jN < n \le (j+1)N \\ \left(\frac{s+r}{n+1}\right)^{\beta} \lambda_1 p; & rN < n \le K \end{cases} \quad \ldots (1)$$

And in the finite population model (FPM), the arrival rate is given by: -

$$\lambda(n) = \begin{cases} (M-n)(\lambda_1 + \lambda_2)p; & 0 \le n \le s \\ (M-n)\left(\frac{s}{n+1}\right)^{\beta} \lambda_1 p; & s < n \le N \\ (M-n)\left(\frac{s+j}{n+1}\right)^{\beta} \lambda_1 p; & jN < n \le (j+1)N \\ (M-n)\left(\frac{s+r}{n+1}\right)^{\beta} \lambda_1 p; & rN < n \le M \end{cases} \quad \ldots (2)$$

The state-dependent service rate for both FCM and FPM is given by: -

$$\mu_n = \begin{cases} s\mu; & 0 \le n \le s \\ s\mu + (n-s)\alpha; & s < n \le N \\ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(n - \overline{s+j}\right)\alpha; & jN < n \le (j+1)N \\ \left(s\mu + \sum_{i=1}^{r} \mu_i\right) + \left(n - \overline{s+r}\right)\alpha; & rN < n \le L \end{cases} \quad \ldots (3)$$

III. QUEUE SIZE DISTRIBUTION

In this section we give the mathematical expressions for the queue size distribution and average queue length for both F.C.M and F.P.M models by employing product type solution.

3.1 Finite Capacity Model (FCM)

By using product type solution the steady state probability P_n is given as

$$P_n = \begin{cases} \dfrac{\rho^n}{n!} p_0; & 0 \le n \le s \\ \dfrac{\rho^s \left(s^\beta\right)^{n-s} (\lambda_1 p)^{n-s} p_0}{(s!)^{1-\beta} (n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)}; & s \le n \le N \\ \dfrac{\rho^s \left(s^\beta\right)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \left\{(s+i)^\beta\right\}^N \left\{(s+j)^\beta\right\}^{n-jN} p_0}{(s!)^{1-\beta} (n!)^\beta \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu = \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=jN+1}^{n} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+j}\right)\alpha\right\}}; & jN < n \le (j+1)N \\ \dfrac{\rho^s \left(s^\beta\right)^{N-s} \prod_{i=1}^{r-1} \left\{(s+i)^\beta\right\}^N \left\{(s+r)^\beta\right\}^{n-rn} p_0}{(s!)^{1-\beta} (n!)^\beta \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu = \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=rN+1}^{n} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+r}\right)\alpha\right\}}; & rN < n \le K \end{cases} \quad \text{…. (4)}$$

Using normalizing condition $\sum_{n=0}^{N} P_n = 1$, we get P_0 as

$$P_0 = \Bigg[\sum_{n=0}^{s} \frac{\rho^n}{n!} + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=s+1}^{N} \frac{\left(s^\beta\right)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)}$$
$$+ \frac{\rho^s}{(s!)^{1-\beta}} \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} \frac{\left(s^\beta\right)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \left\{(s+i)^\beta\right\}^N \left\{(s+j)^\beta\right\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=jN+1}^{n} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+j}\right)\alpha\right\}}$$
$$+ \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=rN+1}^{K} \frac{\left(s^\beta\right)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{r-1} \left\{(s+i)^\beta\right\}^N \left\{(s+j)^\beta\right\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu + \sum_{i=1}^{r} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=rN=1}^{n} \left\{\left(s\mu + \sum_{i=1}^{r} \mu_i\right) + \left(i - \overline{s+r}\right)\alpha\right\}} \Bigg]^{-1} \quad \text{…. (5)}$$

The average number of customers in the system is given by

$$L = \sum_{n=0}^{N} n P_n$$

$$= \Bigg[\sum_{n=0}^{s} \frac{\rho^n}{(n-1)!} + s \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=s+1}^{N} \frac{\left(s^\beta\right)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)}$$
$$+ \frac{\rho^s}{(s!)^{1-\beta}} \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} (s+j) \frac{\left(s^\beta\right)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \left\{(s+i)^\beta\right\}^N \left\{(s+j)^\beta\right\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=jN+1}^{n} \left\{\left(s\mu + \sum_{i=1}^{j} \mu_i\right) + \left(i - \overline{s+j}\right)\alpha\right\}}$$
$$+ \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=rN+1}^{K} (s+r) \frac{\left(s^\beta\right)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{r-1} \left\{(s+i)^\beta\right\}^N \left\{(s+j)^\beta\right\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{\left(s\mu + \sum_{i=1}^{r} \mu_i\right) + \left(i - \overline{s+l}\right)\alpha\right\} \prod_{i=rN=1}^{n} \left\{\left(s\mu + \sum_{i=1}^{r} \mu_i\right) + \left(i - \overline{s+r}\right)\alpha\right\}} \Bigg] P_0 \quad \text{…. (6)}$$

3.2 Finite Population Model (FPM): Substituting the appropriate arrival and service rates in product type solution, the steady state queue size distribution P_n for finite population model is given by

$$P_n = \begin{cases} \dfrac{M!}{(M-n)!}\dfrac{\rho^n}{n!}p_0; & 0 \le n \le s \\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s\left(s^\beta\right)^{n-s}(\lambda_1 p)^{n-s}p_0}{(s!)^{1-\beta}(n!)^\beta\prod\limits_{i=s+1}^{n}(s\mu+(i-s)\alpha)}; & s \le n \le N \\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s\left(s^\beta\right)^{N-s}(\lambda_1 p)^{n-s}\prod\limits_{i=1}^{j-1}\left\{(s+i)^\beta\right\}^N\left\{(s+j)^\beta\right\}^{n-jN}p_0}{(s!)^{1-\beta}(n!)^\beta\prod\limits_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod\limits_{l=1}^{j-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=jN+1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}; & jN < n \le (j+1)N \\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s\left(s^\beta\right)^{N-s}\prod\limits_{i=1}^{r-11}\left\{(s+i)^\beta\right\}^N\left\{(s+r)^\beta\right\}^{n-rn}p_0}{(s!)^{1-\beta}(n!)^\beta\prod\limits_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod\limits_{l=1}^{r-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=rN+1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}; & rN < n \le M \end{cases} \quad \text{…. (7)}$$

In this case, P_0 is also obtained by using normalizing condition as follows:

$$\begin{aligned} P_0 = \Bigg[& \sum_{n=0}^{s}\frac{M!}{(M-n)!}\frac{\rho^n}{n!} + \frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}\frac{M!}{(M-n)!}\frac{\left(s^\beta\right)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{n}(s\mu+(i-s)\alpha)} \\ & + \frac{\rho^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\frac{M!}{(M-n)!}\frac{\left(s^\beta\right)^{N-s}(\lambda_1 p)^{n-s}\prod\limits_{i=1}^{j-1}\left\{(s+i)^\beta\right\}^N\left\{(s+j)^\beta\right\}^{n-jN}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod\limits_{l=1}^{j-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=jN+1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}} \\ & + \frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=rN+1}^{M}\frac{M!}{(M-n)!}\frac{\left(s^\beta\right)^{N-s}(\lambda_1 p)^{n-s}\prod\limits_{i=1}^{r-1}\left\{(s+i)^\beta\right\}^N\left\{(s+j)^\beta\right\}^{n-jN}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod\limits_{l=1}^{r-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum\limits_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=rN=1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\Bigg]^{-1} \end{aligned} \quad \text{…. (8)}$$

We obtain the average number of customers in the system given as

$$\begin{aligned} L = \Bigg[& \sum_{n=0}^{s}\frac{M\,!}{(M-n)!}\frac{\rho^n}{(n-1)!} + \frac{\rho^s}{(s!)^{1-\beta}}\,s\sum_{n=s+1}^{N}\frac{M\,!}{(M-n)!}\frac{\left(s^\beta\right)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{n}(s\mu+(i-s)\alpha)} \\ & + \frac{\rho^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}(s+j)\frac{M\,!}{(M-n)!}\frac{\left(s^\beta\right)^{N-s}(\lambda_1 p)^{n-s}\prod\limits_{i=1}^{j-1}\left\{(s+i)^\beta\right\}^N\left\{(s+j)^\beta\right\}^{n-jN}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod\limits_{l=1}^{j-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=jN+1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}} \\ & + \frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=rN+1}^{M}(s+r)\frac{M\,!}{(M-n)!}\frac{\left(s^\beta\right)^{N-s}(\lambda_1 p)^{n-s}\prod\limits_{i=1}^{r-1}\left\{(s+i)^\beta\right\}^N\left\{(s+j)^\beta\right\}^{n-jN}p_0}{(n!)^\beta\prod\limits_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod\limits_{l=1}^{r-1}\prod\limits_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum\limits_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod\limits_{i=rN=1}^{n}\left\{\left(s\mu+\sum\limits_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\Bigg]P_0 \end{aligned} \quad \text{…. (9)}$$

IV. THE EXPECTED WAITING TIME

E (W_A) and E (W_B) are the expected waiting time for two type of customers A and B, respectively. The expected waiting time E (W) for the current customer in the system will be expressed as follows: -

$$E(W) = (1-p)E(W_A) + pE(W_B)$$

Now we construct the formula of expected waiting time for two type of customers A and B respectively.

4.1 Expected waiting time for Finite capacity model (FCM)

The expected waiting time of type A customers is derived as

$$\begin{aligned}\mu E(W_A) = \Bigg[& \sum_{n=0}^{s} a_n \frac{\rho^n}{n!} + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=s+1}^{N} \left\{ \frac{n-s+1}{s} + a_{s-1} \right\} \frac{(s^\beta)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)} \\ & + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} \left\{ \frac{n-(s+j)+1}{(s+j)} + a_{s+j-1} \right\} \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu+(i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+j})\alpha \right\}} \\ & + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=rN+1}^{K} \left\{ \frac{n-(s+r)+1}{(s+r)} + a_{s+r} \right\} \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{r-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu+(i-s)\alpha) \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i\right) + (i - \overline{s+l})\alpha \right\} \prod_{i=rN=1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i\right) + (i - \overline{s+r})\alpha \right\}} \Bigg] P_0 \end{aligned} \quad \text{…. (10)}$$

Similarly the expected waiting time of type B customer is derived as

$$\begin{aligned}\mu E(W_B) = \Bigg[& \sum_{n=0}^{s} a_{n+1} \frac{\rho^n}{n!} + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=s+1}^{N} \left\{ \frac{n-s+1}{s} + a_{s} \right\} \frac{(s^\beta)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)} \\ & + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} \left\{ \frac{n-(s+j)+1}{(s+j)} + a_{s+j} \right\} \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu+(i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+j})\alpha \right\}} \\ & + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=rN+1}^{K} \left\{ \frac{n-(s+r)+1}{(s+r)} + a_{s+r} \right\} \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{r-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu+(i-s)\alpha) \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i\right) + (i - \overline{s+l})\alpha \right\} \prod_{i=rN=1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i\right) + (i - \overline{s+r})\alpha \right\}} \Bigg] P_0 \end{aligned} \quad \text{…. (11)}$$

The difference (D) between mean waiting times for both types of customers is written as

$$\begin{aligned} D = \mu P_0 \Bigg[& \sum_{n=0}^{s} (a_{n+1} - a_n) \frac{\rho^n}{n!} + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{n=s+1}^{N} (a_s - a_{s-1}) \frac{(s^\beta)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!)^\beta \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)} \\ & + \frac{\rho^s}{(s!)^{1-\beta}} \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} (a_{s+j} - a_{s+j-1}) \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!)^\beta \prod_{i=s+1}^{N} (s\mu+(i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i\right) + (i - \overline{s+j})\alpha \right\}} \Bigg] \end{aligned} \quad \text{…. (12)}$$

4.2 Expected waiting time for finite population model (FPM)

In this model the expected waiting time of type A customer is derived as

$$\mu E(W_A)=\Bigg[\sum_{n=0}^{s}a_n\frac{M!}{(M-n)!}\frac{\rho^n}{n!}+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_{s-1}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}$$
$$+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j-1}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}$$
$$+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\Bigg]P_0 \quad \text{…. (13)}$$

Similarly the expected waiting time of type B customer is derived as

$$\mu E(W_B)=\Bigg[\sum_{n=0}^{s}a_{n+1}\frac{M!}{(M-n)!}\frac{\rho^n}{n!}+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_{s}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}$$
$$+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}$$
$$+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\Bigg]P_0 \quad \text{…. (14)}$$

The difference (D) is obtained by using the both expected waiting time for both types of the customers in this model

$$D=\mu P_0\Bigg[\sum_{n=0}^{s}(a_{n+1}-a_n)\frac{M!}{(M-n)!}\frac{\rho^n}{n!}+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}(a_s-a_{s-1})\frac{M!}{(M-n)!}\frac{(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}$$
$$+\frac{\rho^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}(a_{s+j}-a_{s+j-1})\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}\Bigg] \quad \text{…. (15)}$$

V. SOME SPECIAL CASES

Now we discuss some special cases, which are deduced from analytical results derived in previous sections

Case I: Multi server queue with discouragement, nopassing and additional server

In this case we are assuming only delay customers so that by setting λ_2=0 in performance metrics, we deduce the following:

(a) Finite capacity model: In this model the arrival rate reduces to

$$\lambda(n)=\begin{cases}\lambda_1 p; & 0\le n\le s\\ \left(\frac{s}{n+1}\right)^\beta\lambda_1 p; & s<n\le N\\ \left(\frac{s+j}{n+1}\right)^\beta\lambda_1\ p; & jN<n\le(j+1)N\\ \left(\frac{s+r}{n+1}\right)^\beta\lambda_1 p; & rN<n\le K\end{cases} \quad \text{…. (16)}$$

The queue size distribution is given by

$$P_n=\begin{cases}\dfrac{\rho_1^n}{n!}p_0; & 0\le n\le s\\ \dfrac{\rho_1^s(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}; & s\le n\le N\\ \dfrac{\rho_1^s(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+j})\alpha\right\}}; & jN<n\le(j+1)N\\ \dfrac{\rho_1^s(s^\beta)^{N-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+r)^\beta\}^{n-rn}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=rN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+r})\alpha\right\}}; & rN<n\le K\end{cases} \quad \text{…. (17)}$$

and the expected waiting of type B customer in the model is written as

$$\mu E(W_B)=\left[\sum_{n=0}^{s}a_{n+1}\frac{\rho_1^n}{n!}+\frac{\rho_1^n}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}\right. \quad \text{…. (18)}$$

$$+\frac{\rho_1^n}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+j})\alpha\right\}}$$

$$\left.+\frac{\rho_1^n}{(s!)^{1-\beta}}\sum_{n=rN+1}^{K}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+(i-\overline{s+r})\alpha\right\}}\right]P_0$$

(b) Finite population model: In this model the arrival rate becomes

$$\lambda_n=\begin{cases}(M-n)\lambda_1 p; & 0\le n\le s\\ (M-n)\left(\dfrac{s}{n+1}\right)^\beta\lambda_1 p; & s<n\le N\\ (M-n)\left(\dfrac{s+j}{n+1}\right)^\beta\lambda_1\ p; & jN<n\le(j+1)N\\ (M-n)\left(\dfrac{s+r}{n+1}\right)^\beta\lambda_1 p; & rN<n\le K\end{cases} \quad \text{…. (19)}$$

and the queue size distribution is given by

$$P_n=\begin{cases}\dfrac{M!}{M-n!}\dfrac{\rho_1^n}{n!}p_0; & 0\le n\le s\\ \dfrac{M!}{M-n!}\dfrac{\rho_1{}^s(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}; & s\le n\le N\\ \dfrac{M!}{M-n!}\dfrac{\rho_1{}^s(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+j})\alpha\right\}}; & jN<n\le(j+1)N\\ \dfrac{M!}{M-n!}\dfrac{\rho_1{}^s(s^\beta)^{N-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+r)^\beta\}^{n-rn}p_0}{(s!)^{1-\beta}(n!)^\beta\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=rN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+r})\alpha\right\}}; & rN<n\le K\end{cases} \quad \text{…. (20)}$$

and the expected waiting of type B customer in the model is written as

$$\mu E(W_B)=\Bigg[\sum_{n=0}^{s} a_{n+1}\frac{M!}{(M-n)!}\frac{\rho_1^n}{n!}+\frac{\rho_1^s}{(s!)^{1-\beta}}\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)^\beta\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}$$

$$+\frac{\rho_1^s}{(s!)^{1-\beta}}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}$$

$$+\frac{\rho_1^s}{(s!)^{1-\beta}}\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)^\beta\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\Bigg]P_0 \quad \text{…. (21)}$$

Case II: Multi server loss and delay queue with discouragement, nopassing and additional server

In this case we set β=1 in results derived in previous sections so as to consider liner balking factor. Other assumptions are same. Now we obtain various formulae as follow:

(a) Finite capacity model: In this model the arrival rate reduces to

$$\lambda(n)=\begin{cases}(\lambda_1+\lambda_2)p; & 0\le n\le s\\ \left(\frac{s}{n+1}\right)\lambda_1 p; & s<n\le N\\ \left(\frac{s+j}{n+1}\right)\lambda_1 p; & jN<n\le(j+1)N\\ \left(\frac{s+r}{n+1}\right)\lambda_1 p; & rN<n\le K\end{cases} \quad \text{…. (22)}$$

and the queue size distribution is given by

$$P_n=\begin{cases}\frac{\rho^n}{n!}p_0; & 0\le n\le s\\ \frac{\rho^s(s)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}; & s\le n\le N\\ \frac{\rho^s(s)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}; & jN<n\le(j+1)N\\ \frac{\rho^s(s)^{N-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+r)^\beta\}^{n-rn}p_0}{(n!)\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}; & rN<n\le K\end{cases} \quad \text{…. (23)}$$

And the expected waiting of type B customer in the model is written as

$$\mu E(W_B)=\left[\sum_{n=0}^{s}a_{n+1}\frac{\rho^n}{n!}+\rho^s\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{(s)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}\right. \quad \text{…. (24)}$$

$$+\rho^s\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{(s)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)\}^N\{(s+j)\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}$$

$$\left.+\rho^s\sum_{n=rN+1}^{K}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\right]P_0$$

(b) Finite population model: The arrival rate becomes

$$\lambda_n=\begin{cases}(M-n)(\lambda_1+\lambda_2)p; & 0\le n\le s\\ (M-n)\left(\dfrac{s}{n+1}\right)\lambda\ p; & s<n\le N\\ (M-n)\left(\dfrac{s+j}{n+1}\right)\lambda_1\ p; & jN<n\le(j+1)N\\ (M-n)\left(\dfrac{s+r}{n+1}\right)\lambda_1 p & rN<n\le M\end{cases} \quad \text{…. (25)}$$

The queue size distribution is given by

$$P_n=\begin{cases}\dfrac{M!}{(M-n)!}\dfrac{\rho^n}{n!}p_0; & 0\le n\le s\\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s(s)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}; & s\le n\le N\\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s(s)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}; & jN<n\le(j+1)N\\ \dfrac{M!}{(M-n)!}\dfrac{\rho^s(s)^{N-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+r)^\beta\}^{n-rn}p_0}{(n!)\prod_{i=s+1}^{N}\{s\mu+(i-s)\alpha\}\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu=\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}; & rN<n\le M\end{cases} \quad \text{…. (26)}$$

The expected waiting time of type B customer is given as

$$\mu E(W_B)=\left[\sum_{n=0}^{s}a_{n+1}\frac{M!}{(M-n)!}\frac{\rho^n}{n!}+\rho^s\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{M!}{(M-n)!}\frac{(s)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)}\right. \quad \text{…. (27)}$$

$$+\rho^s\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{M!}{(M-n)!}\frac{(s)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)\}^N\{(s+j)\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+\left(i-\overline{s+j}\right)\alpha\right\}}$$

$$\left.+\rho^s\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+l}\right)\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+\left(i-\overline{s+r}\right)\alpha\right\}}\right]P_0$$

Case III: Multi server loss and delay queue with nopassing and additional server

When the system is without discouragement i.e. $\beta = 0,\ and\, \alpha = 0$, then the arrival rates for FCM and FPM are reduced to

Finite capacity model (FCM):

$$\lambda(n) = \begin{cases} (\lambda_1 + \lambda_2)p; & 0 \le n \le s \\ \lambda_1 p; & s \le n < K \end{cases} \quad \text{…. (28)}$$

Finite population model (FPM):

$$\lambda(n) = \begin{cases} (M-n)(\lambda_1 + \lambda_2)p; & 0 \le n \le s \\ (M-n)\left(\frac{s}{n+1}\right)^{\beta} \lambda_1 p; & s \le n < K \end{cases} \quad \text{…. (29)}$$

In this case, the service rate for both the models is reduced to

$$\mu(n) = \begin{cases} n\mu; & 0 \le n \le s \\ s\mu; & s < n \le N \\ \left(s\mu + \sum_{i=1}^{j} \mu_i\right); & jN < n \le (j+1)N \\ \left(s\mu + \sum_{i=1}^{r} \mu_i\right); & rN < n \le K \end{cases} \quad \text{…. (30)}$$

Now for FCM, the queue size distribution is given by

$$P_n = \begin{cases} \frac{\rho^n}{n!} p_0; & 0 \le n \le s \\ \frac{\rho^s (\lambda_1 p)^{n-s} p_0}{(s!)(s\mu)^{n-s}}; & s \le n \le N \\ \frac{\rho^s (\lambda_1 p)^{n-s} P_0}{(s!)(s\mu)^{n-s}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)^{(j-1)N} \prod_{i=jN+1}^{n}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)}; & jN < n \le (j+1)N \\ \frac{\rho^s (\lambda_1 p)^{n-s} P_0}{(s!)(s\mu)^{n-s}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)^{(r-1)N} \prod_{i=rN+1}^{n}\left(s\mu + \sum_{i=1}^{r}\mu_i\right)}; & rN < n \le K \end{cases} \quad \text{…. (31)}$$

Also the expected waiting of type B customer is reduced to

$$\mu E(W_B) = \Bigg[\sum_{n=0}^{s} a_{n+1}\frac{\rho^n}{n!} + \frac{\rho^s}{(s!)}\sum_{n=s+1}^{K}\left\{\frac{n-s+1}{s} + a_s\right\}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{n-s}}$$
$$+ \frac{\rho^s}{(s!)}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)} + a_{s+j}\right\}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{N-s}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)^{(j-1)N}\prod_{i=jN+1}^{n}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)}$$
$$+ \frac{\rho^s}{(s!)}\sum_{n=rN+1}^{K}\left\{\frac{n-(s+r)+1}{(s+r)} + a_{s+r}\right\}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{N-s}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)^{(j-1)N}\prod_{i=jN+1}^{n}\left(s\mu + \sum_{i=1}^{j}\mu_i\right)}\Bigg]P_0 \quad \text{…. (32)}$$

For FPM, the queue size distribution is given by

$$P_n = \begin{cases} \frac{M!}{(M-n)!}\frac{\rho^n}{n!}p_0; & 0 \le n \le s \\ \frac{M!}{(M-n)!}\frac{\rho^s(\lambda_1 p)^{n-s}p_0}{(s!)(s\mu)^{n-s}}; & s \le n \le N \\ \frac{M!}{(M-n)!}\frac{\rho^s(\lambda_1 p)^{n-s}P_0}{(s!)(s\mu)^{n-s}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)^{(j-1)N}\prod_{i=jN+1}^{n}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)}; & jN < n \le (j+1)N \\ \frac{M!}{(M-n)!}\frac{\rho^s(\lambda_1 p)^{n-s}P_0}{(s!)(s\mu)^{n-s}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)^{(r-1)N}\prod_{i=rN+1}^{n}\left(s\mu+\sum_{i=1}^{r}\mu_i\right)}; & rN < n \le M \end{cases} \quad \text{…. (33)}$$

The expected waiting of type B customer in the model can be written as

$$\mu E(W_B) = \left[\sum_{n=0}^{s} a_{n+1}\frac{M!}{(M-n)!}\frac{\rho^n}{n!} + \frac{\rho^s}{(s!)}\sum_{n=s+1}^{K}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{M!}{(M-n)!}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{n-s}}\right.$$

$$+\frac{\rho^s}{(s!)}\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{M!}{(M-n)!}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{N-s}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)^{(j-1)N}\prod_{i=jN+1}^{n}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)} \quad \text{…. (34)}$$

$$\left.+\frac{\rho^s}{(s!)}\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(\lambda_1 p)^{n-s}}{(s\mu)^{N-s}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)^{(r-1)N}\prod_{i=jN+1}^{n}\left(s\mu+\sum_{i=1}^{j}\mu_i\right)}\right]P_0$$

Case IV: Multi server queue with discouragement nopassing, and additional server.

When the system deals with additional server with discouragement i.e. $\lambda_2 = 0\, and\, \beta = 1$, then we obtain performance indices as follow:

$$\lambda(n) = \begin{cases} \lambda_1 P; & 0 \le n \le s \\ \left(\frac{s}{n+1}\right)\lambda_1 p; & s < n \le N \\ \left(\frac{s+j}{n+1}\right)\lambda_1 p; & jN < n \le (j+1)N \\ \left(\frac{s+r}{n+1}\right)\lambda_1 p; & rN < n \le K \end{cases} \quad \text{…. (35)}$$

and the queue size distribution is given by

(a) Finite capacity model: In this case the arrival rates for FCM is

$$P_n = \begin{cases} \dfrac{\rho_1^n}{n!} p_0; & 0 \le n \le s \\ \dfrac{\rho_1^s (s)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!) \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)}; & s \le n \le N \\ \dfrac{\rho_1^s (s)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!) \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu = \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+j})\alpha \right\}}; & jN < n \le (j+1)N \\ \dfrac{\rho_1^s (s)^{N-s} \prod_{i=1}^{r-1} \{(s+i)^\beta\}^N \{(s+r)^\beta\}^{n-rn} p_0}{(n!) \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu = \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=rN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+r})\alpha \right\}}; & rN < n \le K \end{cases} \quad \text{…. (36)}$$

The expected waiting of type B customer in the model is written as

$$\mu E(W_B) = \Bigg[\sum_{n=0}^{s} a_{n+1} \frac{\rho_1^n}{n!} + \rho_1^s \sum_{n=s+1}^{N} \left\{ \frac{n-s+1}{s} + a_s \right\} \frac{(s)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!) \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)} \quad \text{…. (37)}$$

$$+ \rho_1^s \sum_{i=1}^{r-1} \sum_{jN+1}^{(j+1)N} \left\{ \frac{n-(s+j)+1}{(s+j)} + a_{s+j} \right\} \frac{(s)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)\}^N \{(s+j)\}^{n-jN} p_0}{(n!) \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+j})\alpha \right\}}$$

$$+ \rho_1^s \sum_{n=rN+1}^{K} \left\{ \frac{n-(s+r)+1}{(s+r)} + a_{s+r} \right\} \frac{(s^\beta)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{r-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!) \prod_{i=s+1}^{N} (s\mu + (i-s)\alpha) \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=rN=1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{r} \mu_i \right) + (i - \overline{s+r})\alpha \right\}} \Bigg] P_0$$

(b) Finite population model: For FPM, the arrival rate for this case becomes

$$\lambda(n) = \begin{cases} (M-n)\lambda_1 p; & 0 \le n \le s \\ (M-n)\left(\dfrac{s}{n+1} \right) \lambda_1 p; & s < n \le N \\ (M-n)\left(\dfrac{s+j}{n+1} \right) \lambda_1 p; & jN < n \le (j+1)N \\ (M-n)\left(\dfrac{s+r}{n+1} \right) \lambda_1 p; & rN < n \le M \end{cases} \quad \text{…. (38)}$$

And the queue size distribution is given by

$$P_n = \begin{cases} \dfrac{M!}{(M-n)!} \dfrac{\rho_1^n}{n!} p_0; & 0 \le n \le s \\ \dfrac{M!}{(M-n)!} \dfrac{\rho_1^s (s)^{n-s} (\lambda_1 p)^{n-s} p_0}{(n!) \prod_{i=s+1}^{n} (s\mu + (i-s)\alpha)}; & s \le n \le N \\ \dfrac{M!}{(M-n)!} \dfrac{\rho_1^n (s)^{N-s} (\lambda_1 p)^{n-s} \prod_{i=1}^{j-1} \{(s+i)^\beta\}^N \{(s+j)^\beta\}^{n-jN} p_0}{(n!) \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{j-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu = \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=jN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+j})\alpha \right\}}; & jN < n \le (j+1)N \\ \dfrac{M!}{(M-n)!} \dfrac{\rho_1^s (s)^{N-s} \prod_{i=1}^{r-1} \{(s+i)^\beta\}^N \{(s+r)^\beta\}^{n-rn} p_0}{(n!) \prod_{i=s+1}^{N} \{s\mu + (i-s)\alpha\} \prod_{l=1}^{r-1} \prod_{i=lN+1}^{(l+1)N} \left\{ \left(s\mu = \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+l})\alpha \right\} \prod_{i=rN+1}^{n} \left\{ \left(s\mu + \sum_{i=1}^{j} \mu_i \right) + (i - \overline{s+r})\alpha \right\}}; & rN < n \le M \end{cases} \quad \text{…. (39)}$$

The expected waiting time of type B customer is given as

$$\mu E(W_B)=\Bigg[\sum_{n=0}^{s} a_{n+1}\frac{M!}{(M-n)!}\frac{\lambda_1^n}{n!}+\lambda_1^s\sum_{n=s+1}^{N}\left\{\frac{n-s+1}{s}+a_s\right\}\frac{M!}{(M-n)!}\frac{(s)^{n-s}(\lambda_1 p)^{n-s}p_0}{(n!)\prod_{i=s+1}^{n}(s\mu+(i-s)\alpha)} \quad \text{.... (40)}$$

$$+\lambda_1\sum_{i=1}^{r-1}\sum_{jN+1}^{(j+1)N}\left\{\frac{n-(s+j)+1}{(s+j)}+a_{s+j}\right\}\frac{M!}{(M-n)!}\frac{(s)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{j-1}\{(s+i)\}^N\{(s+j)\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{j-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=jN+1}^{n}\left\{\left(s\mu+\sum_{i=1}^{j}\mu_i\right)+(i-\overline{s+j})\alpha\right\}}$$

$$+\lambda_1\sum_{n=rN+1}^{M}\left\{\frac{n-(s+r)+1}{(s+r)}+a_{s+r}\right\}\frac{M!}{(M-n)!}\frac{(s^\beta)^{N-s}(\lambda_1 p)^{n-s}\prod_{i=1}^{r-1}\{(s+i)^\beta\}^N\{(s+j)^\beta\}^{n-jN}p_0}{(n!)\prod_{i=s+1}^{N}(s\mu+(i-s)\alpha)\prod_{l=1}^{r-1}\prod_{i=lN+1}^{(l+1)N}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+(i-\overline{s+l})\alpha\right\}\prod_{i=rN=1}^{n}\left\{\left(s\mu+\sum_{i=1}^{r}\mu_i\right)+(i-\overline{s+r})\alpha\right\}}\Bigg]P_0$$

Case V: In case, when the assumptions of loss customers, discouragement and additional servers are not considered i.e. $\lambda_2=0$, $\alpha=0$, $\beta=0$ and r = 0 then our result tally with those of Sharma et al. (1983).

Case VI: In case $\lambda_1=\lambda$, $\lambda_2=0$, $\alpha=0$, $\beta=0$ and r=0 then our model coincide with the model of Washburn (1974).

VI. NUMERICAL RESULTS

For finite capacity models, the numerical results for performance indices, which are derived in the previous sections, are obtained for illustrative purposes. From these numerical results, we examine the effect of various parameters on system characterization. The numerical results for the difference (D) between the expected waiting times of both type of customers are arranged in tables 1-4. In figures 1-4, we depict the effect of variation of parameters on queue length graphically.

Tables 1-4, display the numerical results for the differences (D) between the expected waiting times of two type of customers by varying the arrival rate of customers. In table 1, we display the difference (D) between the expected waiting times of two type of customers for different values of s by fixing $\mu=1$, $\rho=3$, N=2, $\alpha=0.5$, n=30, $\lambda_1=0.5$ and $\lambda_2=0.5\lambda_1$. Table 2 depicts the results for difference (D) for fixed parameters $\mu=2$, N=2, $\alpha=0.5$, s=1, n=20, $\lambda_1=0.5$ and $\lambda_2=0.2\lambda_1$ by varying r. By fixing $\mu=2$, $\rho=3$, N=2, $\alpha=0.5$, s=1, n=20, $\lambda_1=0.5$ and $\lambda_2=0.5\lambda_1$, the differences (D) for different values of N are given in table 3. In table 4, for fixed parameters $\mu=2$, $\rho=3$, N=2, $\alpha=0.5$, s=1, n=20, and $\lambda_1=0.5$, we present the results for difference (D) between the expected waiting times of both type of customers for different values of ρ_2. It is noted from these tables that difference (D) increases with s, N and ρ_2 whereas decreases with λ.

From figure 4.1, we see that the average number of customers increases by increasing the traffic intensity of delay customers and number of parameters servers, however as traffic intensity crosses the ρ_1 =1.8, the average queue length become almost constant. In figures 4.2 and 4.3, the average numbers of customers increases for increasing value of r and N respectively. Figure 4.4 also indicates the increment in the average number of customers for the increasing value of both traffic intensities ρ_2 and ρ_1.

From the sensitive analysis carried out by varying different parameters, we conclude that

- The average number of customers in the system and expected waiting time increase as traffic intensity of both loss and delay customer's increases, which is obvious.
- As the average number of customers and expected waiting time decrease with the increases in population size, however incorporation of additional servers shows the reverse trend; this is due to the fact that the adding additional servers are facilitated one by one as queue size grows according to threshold level policy.

VII. DISCUSSION

In this investigation, we have developed a loss and delay queueing model with nopassing, additional removable servers and discouragement. In order to check the discouraging behaviour of the customers, the provision of additional servers may be a cost effective alternative in particular when nopassing constraint is imposed.

The essential formulae for steady state queue size distribution, expected waiting time of both type of customers, average number of the customers in the system are derived in explicit form which are computationally tactible as shown by numerical illustration.

λ_1	s		
	1	2	3
0.5	0.520781	0.557776	0.568604
1.0	0.228702	0.248585	0.278192
1.5	0.086558	0.083861	0.102178
2.0	0.032077	0.026176	0.033215

2.5	0.012071	0.008137	0.010397
3.0	0.004637	0.00256	0.003208
3.5	0.001816	0.000818	0.000983
4.0	0.000724	0.000266	0.000301

Table 1: Difference between expected waiting times of both type of customers.

λ_1	r		
	1	2	3
0.2	0.467838	0.460206	0.460441
0.4	0.451492	0.430649	0.431371
0.6	0.438231	0.398626	0.399735
0.8	0.427258	0.364619	0.365471
1.0	0.417887	0.329457	0.328968
1.2	0.409556	0.29416	0.291125
1.4	0.401829	0.259756	0.253217
1.6	0.394373	0.227132	0.216642
1.8	0.386944	0.196945	0.182638
2.0	0.379372	0.169595	0.152072

Table 2: Difference between expected waiting times of both type of customers.

λ_1	N		
	2	4	6
0.5	0.391769	0.394193	0.394487
1.0	0.302482	0.31598	0.319435
1.5	0.211607	0.243517	0.256775

2.0	0.13625	0.18361	0.213042
2.5	0.083451	0.136743	0.184536
3.0	0.05003	0.100704	0.164633
3.5	0.029862	0.073222	0.148304
4.0	0.01789	0.052525	0.132796

Table 3: Difference between expected waiting times of both type of customers.

λ	ρ_2		
	$=0.1\rho_1$	$=0.3\rho_1$	$=0.5\rho_1$
0.5	0.60009	1.003395	1.199319
1.0	0.770292	1.153237	1.271698
1.5	0.980023	1.253214	1.316108
2.0	1.160221	1.317103	1.347984
2.5	1.28401	1.363151	1.37782
3.0	1.366099	1.404014	1.411046
3.5	1.427765	1.445982	1.449516
4.0	1.482099	1.491164	1.493075

Table 4: Difference between expected waiting times of both type of customers.

APPENDIX

The tagged customer has a cumulative distribution function (c. d. f.) of the service times which is given by

F(x) =(I-p) +p(1-expμ_nx)for $x \geq 0,\ 0 < p \leq 1$.

The expression for the mean waiting times of type A and type B customers are given by (cf. Jain and Singh, 2003)

$$E(W_A) = \frac{1}{\mu}\left[\sum_{n=0}^{s} a_n + \sum_{n=s+1}^{K}\left\{\frac{n-s+1}{s} + a_{s-1}\right\} + \sum_{j=1}^{r-1}\sum_{n=jk+1}^{(j+1)K}\left\{\frac{n-(s+j)+1}{(s+j)} + a_{s+j-1}\right\} + \sum_{n=rK+1}^{L}\left\{\frac{n-(s+r)+1}{(s+r)} a_{s+r-1}\right\}\right] P_n \quad \text{…. A.1}$$

and

$$E(W_B) = \frac{1}{\mu}\left[\sum_{n=0}^{s} a_{n+1} + \sum_{n=s+1}^{K}\left\{\frac{n-s+1}{s} + a_{s1}\right\} + \sum_{j=1}^{r-1}\sum_{n=jk+1}^{(j+1)K}\left\{\frac{n-(s+j)+1}{(s+j)} + a_{s+j1}\right\} + \sum_{n=rK+1}^{L}\left\{\frac{n-(s+r)+1}{(s+r)} a_{s+r}\right\}\right] P_n \quad \text{…..A.2}$$

$$\text{Here } a_n = \begin{cases} 0; & n = 0 \\ \sum_{i=1}^{n} 1/i; & n = 1,2,\ldots, n. \end{cases} \quad \text{…. A.3}$$

In the above equations L takes the value N and M in case of finite capacity and finite population models, respectively.

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