Stability of a Quadratic-Additive Functional Equation: A Fixed Point Approach

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Abstract: In this paper, we investigate the stability of a functional equation

\[ f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 2f(x)+2f(y)+2f(z)+f(-x)+f(-y)+f(-z) \]

by using the fixed point theory in the sense of L. Cadariu and V. Radu.

Key words and phrases: Hyers-Ulam stability, fixed point method, a mixed type functional equation.

I. INTRODUCTION

In 1940, S. M. Ulam [14] raised a question concerning the stability of homomorphisms: Given a group G_1, a metric group G_2 with the metric d(.,.), and a positive number ε, does there exist a δ> 0 such that if a mapping f : G_1→G_2 satisfies the inequality

\[ d(f(xy), f(x)f(y)) < δ \]

for all x, y ∈ G_1 then there exists a homomorphism F : G_1→G_2 with

\[ d(f(x), F(x)) < ε \]

for all x ∈ G_1?

When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In the next year, D. H. Hyers[6] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [12] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5], [8]-[10].

Almost all subsequent proofs, in this very active area, have used Hyers' method. Namely, the function F, which is the solution of a functional equation, is explicitly constructed, starting from the given function f, by the formulae

\[ F(x) = \lim_{n→∞} \frac{f^n(x)}{2^n} \text{ or } F(x) = \lim_{n→∞} 2^n f \left( \frac{x}{2^n} \right). \]

We call it a direct method. In 2003, L. Cadariu and V. Radu [2] observed that the existence of the solution F for a functional equation and the estimation of the difference with the given function f can be obtained from the fixed point theory alternative. This method is called fixed point method. In 2004, they [4] applied this method to prove stability theorems of the Cauchy functional equation:

\[ f(x + y) = f(x) + f(y) \quad (1.1) \]

In 2003, they [3] obtained the stability of the quadratic functional equation:

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.2) \]

by using the fixed point method. Notice that if we consider the functions f_1 : ℝ → ℝ defined by f_1(x) = ax and f_2(x) = ax^2, where a is a real constant, then f_1 satisfies the equation (1.1) and f_2 holds (1.2), respectively. We call a solution of (1.1) an additive mapping and a function satisfying (1.2) is called a quadratic map. Now we consider the functional equation:

\[ f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 2f(x)+2f(y)+2f(z)+f(-x)+f(-y)+f(-z) \quad (1.3) \]

which is called a mixed type functional equation. The function f : ℝ→ℝ defined by f(x) = ax^2 + bx satisfies this functional equation, where a, b are real constants. We call a solution of (1.3) a quadratic-additive function. In 2002, S.-M. Jung [7] obtained a stability of a quadratic-additive functional equation by handling the odd part and the even part of the given function f, respectively. In his processing, he needed to take an additive mapping A which is close to the odd part of f and a quadratic mapping Q which is approximate to the even part of it, and then combining A and Q to prove the existence of a quadratic-additive function F which is close to the given function f.

In this paper, we will prove the Hyers-Ulam stability of the quadratic-additive functional equation (1.3) by using the fixed point theory.

II. MAIN RESULTS

We recall the following result of the fixed point theory by Margolis and Diaz.

Theorem 2.1. ([11] or [13]) Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping J : X → X with the Lipschitz constant 0 < L < 1 are given. Then, for each given element x ∈ X, either

\[ d(J^n x, J^{n+1} x) = +∞, \quad ∀n ∈ N \cup \{0\}, \]
or there exists a nonnegative integer k such that:
(1) $d(J^n x; J^{n+1} x) < +\infty$ for all $n \geq k$;
(2) The sequence $\{J^n x\}$ is convergent to a fixed point $y^*$ of $J$;
(3) $y^*$ is the unique fixed point of $Y = \{y \in X, d(f^k x, y) < +\infty\}$;
(4) $d(y, y^*) \leq (\frac{1}{1-\varepsilon})d(y, Jy)$ for all $y \in Y$.

Throughout this paper, let $X$ be a (real or complex) linear space and $Y$ a Banach space. For a given mapping $f : V \rightarrow Y$, we use the following abbreviation

$$Df(x, y, z) := f(x+y+z)+f(y)+f(z)+f(x) - 2f(x)+2f(y)+2f(z) - f(x) - f(y) - f(z)$$

for all $x, y, z \in X$. If $f$ is a solution of the functional equation

$$Df(x, y, z) = 0, \text{ see (1.3)},$$

we call it a quadratic-additive mapping.

**Stability of (1.3)**

In the following theorem, we can prove the stability of the functional equation (1.3) using the fixed point theory.

**Theorem 2.2.**

If $f : X \rightarrow Y$ is a function such that

$$\|Df(x, y, z)\| \leq \varepsilon \quad (3.1)$$

for all $x, y, z \in X$ with a constant $\varepsilon > 0$ and $f(0) = 0$, then there exists a unique quadratic-additive mapping $F : V \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \varepsilon \quad (3.2)$$

for all $x \in X$. In particular, $F$ is represented by

$$F(x) = \lim_{n \to \infty} \left( \frac{f(2^n x) + f(-2^n x)}{2^{n+1}} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right) \quad (3.3)$$

for all $x \in X$.

**Proof**

Let $S$ be the set of all functions $g : X \rightarrow Y$ with $g(0) = 0$. We introduce a generalized metric on $S$ by

$$d(g, h) = \inf \{K \varepsilon \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K \varepsilon \text{ for all } x \in X\}.$$

It is easy to show that $(S, d)$ is a generalized complete metric space. Now we consider the mapping $J : S \rightarrow S$, which is defined by

$$Jg(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $x \in X$. Notice that

$$J^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2^{n+1}}$$

for all $n \in \mathbb{N}$ and $x \in X$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of $d$, we have

$$\|J^n g(x) - J^n h(x)\| \leq \frac{3}{8} \|g(2x) - h(2x)\| + \frac{1}{8} \|g(-2x) - h(-2x)\| \leq \frac{K}{2} \varepsilon$$

for all $x \in X$, which implies that

$$d(Jg, Jh) \leq \frac{1}{2} d(g, h)$$

for any $g, h \in S$. That is, $J$ is a strictly contractive self-mapping of $S$ with the Lipschitz constant $\frac{1}{2}$. Moreover, by (3.1) we see that

$$\|f(x) - Jf(x)\| \leq \frac{1}{8} \|3Df(x, x, -x) - Df(-x, x, x)\| \leq \frac{\varepsilon}{2}$$

for all $x \in X$. It means that $d(f, Jf) \leq \frac{1}{2} < \infty$ by the definition of $d$. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \rightarrow Y$ of $J$ in the set $T = \{g \in S : d(f, g) \leq \infty\}$, which is represented by (3.3) for all $x \in X$. Notice that

$$d(f, F) \leq \frac{1}{1 - \frac{1}{2}} d(f, Jf) \leq \varepsilon$$

Which implies (3.2). By the definition of $F$, together with (3.1) and (3.3), we have

$$\|DF(x, y, z)\| = \lim_{n \to \infty} \left( \frac{Df(2^n x, 2^n y, 2^n z) - Df(-2^n x, -2^n y, -2^n z)}{2^{n+1}} \right)$$

for all $x, y, z \in X$. For a given mapping $f : X \rightarrow Y$, we use the following abbreviations

$$Af(x, y) := f(x + y) - f(x) - f(y), \quad Qf(x, y) := f(x + y) + f(x) - f(2x) - f(2y)$$

for all $x, y \in X$. Using Theorem 3.1, we will show the stability results of the additive functional equation $Af \equiv 0$ and the quadratic functionalequation $Qf \equiv 0$ in the following corollaries.

**Corollary 3.2**

If $f : X \rightarrow Y$ is a function such that

$$\|Af(x, y)\| \leq \varepsilon \quad (3.4)$$

for all $x, y \in X$ with a constant $\varepsilon > 0$ and $f(0) = 0$, then there exists a unique additive mapping $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq 7 \varepsilon \quad (3.5)$$

for all $x \in X$. In particular, the mappings $F$ is represented by
F(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \quad (3.6)

for all x \in X.

**Proof.**

Notice that

$$\|Df(x, y, z)\| = \|Af(x, y + z) - Af(x - y, y - z) - Af(z - x, x - z) + Af(y, z) + Af(x, -x) + Af(y, -y) + Af(z, -z)\| \leq 7\varepsilon$$

for all x, y, z \in X. Therefore, according to Theorem 3.1, there exists a unique mapping F : X \to Y satisfying (3.2), which is represented by (3.3). Observe that

$$\lim_{n \to \infty} \left\| \frac{f(2^n x) + f(-2^n x)}{2^{n+1}} \right\| = \lim_{n \to \infty} \left\| \frac{f(2^n x) + f(-2^n x) - f(0)}{2^n} \right\| = \lim_{n \to \infty} \frac{1}{2^{n+1}} \|Af(2^n x, -2^n x)\| \leq \lim_{n \to \infty} \frac{1}{2^{n+1}} \varepsilon = 0$$

as well as

$$\lim_{n \to \infty} \left\| \frac{f(2^n x) + f(-2^n x)}{2^n} \right\| \leq \lim_{n \to \infty} \frac{\varepsilon}{2^n} = 0$$

for all x \in X. From this and (3.3), we get (3.6). Moreover, we have

$$\|Af(2^n x, 2^n y)\| \leq \frac{\varepsilon}{2^n}$$

for all x, y \in X. Taking the limit as n \to \infty in the above inequality, we get

$$AF(x, y) = 0$$

for all x, y \in X.

**Corollary 2.4.**

If f : X \to Y is a function such that

$$\|Qf(x, y)\| \leq \varepsilon \quad (3.7)$$

for all x, y \in X with a constant \varepsilon > 0 and f(0) = 0, then there exists a unique quadratic mapping F : X \to Y such that

$$\|f(x) - F(x)\| \leq \frac{26}{4} \varepsilon \quad (3.8)$$

for all x \in X. In particular, the mappings F is represented by

$$F(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \quad (3.9)$$

for all x \in X.

**Proof.**

Notice that

$$\|Df(x, y, z)\| = \|Qf(x, y) - Qf(x, z) - Q(y) f(x) - Q(z) f(x) + 4Qf(x, y, z, x) + 4Qf(x, y, z, x) - 4Qf(0, 0, 0, 0)\| \leq 264\varepsilon$$

for all x, y, z \in X. According to Theorem 3.1, there exists a unique mapping F : V \to Y satisfying (3.8) which is represented by (3.3).

Observe that

$$\lim_{n \to \infty} \left\| \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right\| = \lim_{n \to \infty} \|Qf(0, 2^n x)\| \leq \lim_{n \to \infty} \frac{\varepsilon}{2^n} = 0$$

for all x \in X. From this and (3.3), we get (3.9) for all x \in X. Moreover, we have

$$\|Qf(2^n x, 2^n y)\| \leq \frac{\varepsilon}{4^n}$$

for all x, y \in X. Taking the limit as n \to \infty in the above inequality, we get

$$QF(x, y) = 0$$

for all x, y \in X.

**REFERENCES**


