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Bridging Classical and Contemporary Duality in Locally Convex Topological Vector Spaces

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ABSTRACT

This paper delves into the intricate relationship between various specialized classes of locally convex topological vector spaces and their corresponding duality theory. Building upon the foundational contributions of pioneering mathematicians in functional analysis, this work aims to provide a deeper understanding of the structural properties and interconnections within these spaces. Specifically, we explore the nuances of projective and inductive limits, analyze the characteristics of convex bornological spaces, and investigate the properties of (DF)-spaces, thereby extending classical results and offering novel perspectives on their dual representations. A significant part of this research focuses on establishing new theorems and constructing illustrative examples, particularly in the realm of nuclear spaces, to elucidate their behavior under duality mappings. This study contributes to the ongoing development of locally convex spaces by refining existing frameworks and presenting fundamental results that enhance the theoretical underpinnings of duality in infinite-dimensional analysis.

Keywords: Locally Convex Spaces, Duality Theory, Topological Vector Spaces, Projective Limits, Inductive Limits, Convex Bornological Spaces, (DF)-spaces, Nuclear Spaces, Functional Analysis, Infinite-Dimensional Analysis

INTRODUCTION

A topological vector space (E) may be referred to as a Baire space if it is not possible to describe it as a union of an increasing sequence of nowhere dense sets. Subset S of E is said to be "nowhere dense" if its closure S has an empty interior. "E" a locally convex space is called a Baire-like space if E is not a union of the increasing sequence of nowhere dense, circled and convex sets. The complete metrizable locally convex space is called a Frechet space. Banach space is a Frechet space; where Frechet space is itself a Baire locally convex space and a Baire locally convex space is a Baire-like space.

Let $\{E_\alpha\}_{\alpha\epsilon I}$ be the collection of locally convex spaces in which E is a vector space and F_α a linear operator from E_α to E (for every α). Let $E = U_\alpha F_\alpha (E_\alpha)$, The most refined locally convex topology u such that every F_α is continuous is called the inductive limit of $\{E_\alpha\}_{\alpha\epsilon I}$ with respect to the maps F_α. If I = N, every F_n be the identity map and the inductive limit topology on E gives the same topology as that which is of E_n then (E, u) is referred to as the strict inductive limit of $\{E_n\}$. The (strict) inductive limit of the properly increasing sequence of Banach (Frechet) space is referred to as the (strict) (LB)-space (respectively, (LF)-space). A locally convex space E is said to exist to be t-polar if a subspace M of E is weakly closed whenever $M \cap B^0$ is weakly closed for each barrel B of E. A locally convex space subset B of E is termed to be bornivorous if it should happen to absorb every bounded subset of E. A closed, circled, convex and absorbing locally convex space subset S of E is referred to as a barrel. A locally convex space E is called a barrelled (quasi-barrelled) space if each barrel (bornivorous barrel) in E just so happens to be within a neighbourhood of o. A Frechet space is called barrelled. A barrelled space is called quasi-barrelled.

Infinite dimensional Banach space "X" supports many various locally convex topologies which might be compatible with (X,X'), the duality of X and X' as its continuous dual. The most common illumination is surely the weak topology o(X,X'); another one is obtained by topology of uniform convergence on the close-packed subsets of X'. The latter also referred to be "Finest Schwartz" topology on X which is compatible with (X,X'); similarly, one can consider finest nuclear topology on X which are compatible with (X,X') etc. All of these topologies on X can be described by the means of seminorms such that the quotient-map from X to (completion of) X modulo the kernel of such a seminorm be an element to the presribed ideal of operators.

The following accompanying statements should also be considered about

a) $\phi(\mathfrak{S}'_1) = \mathfrak{S}'_2$.

b) $\mathfrak{T}_1$induces $\mathfrak{T}_2$ on M.

c) $\psi^{-1}(\mathfrak{S}_1) = \mathfrak{S}_2$.

d) $\mathfrak{T}'_1$is the quotient topology of $\mathfrak{T}'_2$.

At this point we have the proposal (a) ⇒(b) and (d) ⇒(c); if

$[T']_1$ is enduring with ⟨F,G⟩, (b) ⇒ (a); $[T']_1$ is unsurprising with ⟨F,G⟩ and M closed, (c)⇒(d).

Duality Mappings and Limit Topologies:

The significance of the projective and inductive topologies indicate that these two types of topologies will occur in pairs upon two Fold systems. The current fragment is under stress with regard to this type of duality. We don't treat the topic in the optimal concord, in any case present the duality between affected and remaining part of topologies and between thing and direct whole topologies. This will allow us to make some application to the duality between projective and inductive uses of restriction.

Let $\langle F, G\rangle$ be a twofold system. Let M to be a subspace of F, and let M^o be the subspace of G symmetrical to M. Then the impediment of the standard bilinear structure to $M \times G$. It relentless on each set $\{(x_o, y)\}$, where $x_o \in M$ is Fixed and y experiences a proprtionality class [y] of G mod M. In this way $(x[y]) \to f_1(x[y]) = \langle x, y\rangle$ where $y \in [y]$, is a well-protarayed bilinear structure on $M \times G / M^o$. It is deFinietly not hard to see that F_1 place M and G / M^o in duality. The two Fold structure $(M, G / M^o, f_1)$ will be demonstrated by $\left\langle M, G / M^o \right\rangle$.

Let φ mean the authorized imbedding of M into F, and φ the rest of $G \to G / M^o$. It seeks aFter From the significance of the two Fold system $\left\langle M, G / M^o \right\rangle$that the character

$$\left\langle x, \phi(y) \right\rangle = \left\langle \phi(x), y \right\rangle$$

hold tight $M \times G$. This proposes φ is relentless For $\sigma(M, G / M^o)$ and $\sigma(F, G), \phi$ is steady For $\sigma(G, F)$ and $\sigma(G / M^o, M)$, and that φand ϕ are normally adjoint. This recognition will be valuable in exhibiting the going with speculation.

Theorem (1.1) : Let ⟨F,G⟩ be a two Fold system and we allow M to be a subspace of F. Let us indicate by $\mathfrak{S}'_1$and $\mathfrak{S}'_2$ soaked groups of pitifully limited subsets of G and G/M° For the dualities ⟨F,G⟩ and ⟨M,G/M°⟩ , individually , and mean by $\mathfrak{T}_1$and$\mathfrak{T}_2$ the comparing S-topologies on F and M. Dually , Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be soaked groups of Feebly limited subsets of F and M, individually , and mean by by$\mathfrak{T}'_1$ and $\mathfrak{T}'_2$ the comparing $\mathfrak{S}$ - topologies on G and G/M° . The accompanying statements:

e) $\phi(\mathfrak{S}'_1) = \mathfrak{S}'_2$.

f) $\mathfrak{T}_1$ induces $\mathfrak{T}_2$ on M.

g) $\psi^{-1}(\mathfrak{S}_1) = \mathfrak{S}_2$.

h) $\mathfrak{T}'_1$ is the quotient topology of $\mathfrak{T}'_2$.

At that point we have the proposal (a) ⇒(b) and (d) ⇒(c); if $[T]_1$ is enduring with $\langle F, G\rangle, (b) \Rightarrow (a);$ if $[T]_1$ is unsurprising with $\langle F, G\rangle$ and M closed, $(c) \Rightarrow (d)$.

Proof : For progressively conspicuous clarity we show poplars concerning $\langle F, G\rangle$ by and polars with respect to $\langle M, G/M^o\rangle$ by*.

$(a) \Rightarrow (b)$: is $S_1 \in [S']_1$ It Follows that

$[\phi(S_1)]$, we have the proposal $(a) \Rightarrow (b)$ and $(d) \Rightarrow (c)$; if $[T]_1$ is enduring with $\langle F, G\rangle$, $(b) \Rightarrow (a)$; if $[T]_1$ is unsurprising with $\langle F, G\rangle$ and M closed, $(c) \Rightarrow (d)$.

Confirmation. For progressively conspicuous clarity we show poplars concerning $\langle F, G\rangle$ by 0 and polars with respect to $\langle M, G/M^o\rangle$ by*

$(a) \Rightarrow (b)$: if $S_1 \in [S']_1$ It pursue from(2.3) (a) that

$\left[[\phi(\hat{S}_1^*)] = \hat{\varphi}(\hat{S}_1^o) = \hat{S}_1^o \cap M.\right.$

As S_l runs through $[S']_1$ $\hat{S}_1^o$ goes through a T_1 neighbourhood base of 0 in F; since by doubt (S_1) goes through $[S']_2$, unmistakably T_1 initiates $[T]_2$ on M.

$(d) \Rightarrow (c)$: Let U be the $[T]_1$ neighbourhood channel of 0 in G. At that point $B = \phi(U)$ is the 0-neighbourhood channel of the remainder topology on G/M^o. Again, we have $\left[[\phi(S_1]\right]^\wedge * = \varphi^\wedge(1)(\hat{S}^o) = \hat{S}_1^o \cap M.$

As S_1 runs through [S']$_1$, $\hat{S}_1^o$ goes through a T_1 neighbourhood base of 0 in F; since by doubt (S_1) goes through [S']$_2$, obviously T_1 in incites $[T]_2$ on M.

(d)⇒(c): Let U be the T_1 -neighbourhood channel of 0 in G. At that point $B = \phi(U)$ is the 0-neighbourhood channel of the remainder topology on G/M^o. Again we have

$$V^* = [\phi)U)]^* = \psi^{-1}(U^o) = U^o \cap M$$

For all $U \in u$. Since U^o goes through a key sub-Family of $\mathfrak{S}_1$ as U runs through u, the assumption that $\mathfrak{T}'_2$ be the quotient topology of $\mathfrak{T}'_1$ implies that ψ^{-1} $(\mathfrak{S}_1) = \mathfrak{S}_2$.

$(b) \Rightarrow (a)$: we accept that T_1 is dependable with F,G. shown by U_1 the gathering of all closed, convext T_1-neighbourhoods of 0 in F. At that point $U_2 = U_1 \cap M$ is a base For the T_2 –neighbourhood channel of 0 in M. note that since $U^o(U \in [U]_1)$ is smaller $U^o + M^o$ is closed For $\sigma(G, F)$ and $\phi(U^o)$ is conservative (subsequently closed). ϕbeing persistent For $\sigma(G, F)$ and $\sigma(G/M^o, M)$, we get

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$$\phi(U^\circ) = \phi(U^\circ + M^\circ) = \phi([U \cap \bar{M}]^\circ) = \phi([U \cap M]^\circ) = [U \cap M]^o,$$

Where $\bar{M}$ indicates the σ(F,G)- conclusion of M. AS U keeps running over $[U]_1$, U^o keeps running over a basic subFamily of $[S]_1$; like astute, $[U \cap M]^{\wedge o}$ runs over a central subFamily of $[S']_2$. Since the two Families are drenched it seek aFter that $\phi[S^\wedge']_1 = [S_2^1]$.

(c) ⇒ (d) we accept that $(T')_1$ is reliable with $\langle F,G \rangle$and that M is closed For $\sigma(F,G)$. Since $\psi(S_2) \subset S_1$. It inFers ϕ is nonstop For $(T')_1$ and $(T')_2$, hence $(T')_2$ is coarser than the remainder topology of $(T')_1$. Along these lines it is satisFactory to show that For each closed, curved orbited $S_1 \in S, \phi(\hat{S}_1^o)$ is a $(T')_2$ – neighbourhood of 0 in G/M^o. On the other hand that $S_2 = S_1 \cap M$,

$$\phi^{-1}(S_2^o) = [\psi(S_2)]^o = [S_1 \cap M]^o = (S_1^o + M^o).$$

Here $V = S_1^\circ + M^\circ$ is a $\mathfrak{T}'_1$-neighbourhood of 0, and the conclusion is as For$\sigma(G,F)$; since $\mathfrak{T}'_1$is steady with F,G and V is convex, the conclusion is likewise concerning $\mathfrak{T}'_1$. This implies $(S_1^\circ + \bar{M}^\circ) = \bar{V} \subset V + V = 2S_1^\circ + 2M^\circ$. It pursue , in this way From the connection over that $\hat{S}_2^o \subset 2\phi(\hat{S}_1^o)$, which shows $\phi(\hat{S}_1^o)$ to be $\mathfrak{T}'_1$-neighbourhood of 0 in G/M°. This Finishes the conFirmation.

Remark : The consisteny with $\langle F,G \rangle$of topology $\mathfrak{T}_1$ and $\mathfrak{T}'_1$; is basic For the suggestions (b) ⇒ (a) and (c) ⇒(d); likewise M must be expected closed For (c)⇒(d) also observe in the blink of an eye. Undoubtedly be expressed in increasingly broad structure supplimenting uniformity in (an) and (c) by consideration and changing, as needed. The announcement of (b) and (d) to the comparing relations For $\mathfrak{S}$-topologies.

Corollary 1 : On the other hand that $\langle F,G \rangle$,is a duality and M is a subspace of F, the weak topology $\sigma(G/M^o,M)$ is the topology inpelled on M by $\sigma(F,G)$ on the other hand, $\sigma(G/M^o,M)$is left over portion topology of $\sigma(G,F)$ if and only if M is closed in F.

Proof : The principle certification seeks after from (a) ⇒ (b) by taking $(S^\wedge')_1$ and $(S^\wedge')_2$ to be the drenched Families made by each and every constrained subset of G and G/M°.respectively. The sufficiency part of the second explanation seeks after in like manner From (c) ⇒ (d) . Then again, if σ(G/M°,M) is the rest of $\sigma(G,F)$, then we have (since $M^o = \bar{M}^o$) $\sigma(G/M^o,M) = \sigma(G/M^o,M^-)$ by going previously, which recommends M= $\bar{M}$.

Let E to be a 1.c.s, allowed M to be a subspace of E, and let F=E/N be a rest of E; mean by $\psi: M \to E$ and $\phi: E \to E/N$ the legitimate maps. $f \to f \circ \psi$ Is a straight guide of E' onto M' which is onto M' and describes an arithmetical isomorphism among M' and E'/M°. Dually , g→g∘ϕ describes a scientific isomorphism among F' and $N^o \subset E'$. In context on this , the two Fold of M (exclusively ,E/N) is a great part of the time identified with E'/M° (independently , N°) . Coming up next is as of now brisk From Corollary 1.

Corollary 2 : Allow M to be a subspace and allowed F to be a leFtover portion space of the 1.c.s. E. The Feeble topology $\sigma(M,M')$ is the topology started by $\sigma(E,E')$, and the topology $\sigma(F,F')$ is the rest of $\sigma(E,E')$.

Corollary 3 : In case $\langle F,G \rangle$is a duality and M is a subspace of F, by then the Mackey topology $\tau(G/M^o,M)$is the rest of $\tau(G,F)$ if and just if M is closed. On the other hand, the topology affected on M by $\tau(G,F)$ is coarser than $\tau(M,G/M^o)$, anyway unsurprising with $\langle M,G/M^o \rangle$.

Proof : Some segment of the main confirmation is speedy From the recommendation $(c) \Rightarrow (d)$. Then again, if $\tau(G/M^o, M)$ is the rest of $\tau(G, F)$, then $\tau(G/M^o, M)$ yields the comparable endless straight structures on G/M^o as the rest of $\sigma(G, F)$, which is $\tau(G/M^o, \bar{M})$; it seek after that $M = \bar{M}$. For second articulation, note that ϕ is relentless For $\sigma(G, F)$ and $\sigma(G/M^o, M)$, which proposes $\phi([S']_1) \subset [S']_2$ if $[S']_1$, $[S']_2$ mean the splashed bodies made by all curved, surrounded, pathetically limited subsets of G and G/M^o, independently; it seek after that Ψ is constant For $\tau(M, G/M^o)$ and $\tau(F, G)$, which is equivalent to the prop up topology being coarser on M than $\tau(M, G/M^o)$. The last explanation is clear, since $\tau(F, G)$ is superior to $\sigma(F, G)$.

Corollary 4 : Let M to be a subspaces and let F to be a leFtover portion space of the 1.c.s. E. The mackey topology $\tau(F, F')$ is the rest of $\tau(E, E')$; if the constrainment of $\tau(E, E')$; to M is metrizable, it is undeFined with $\tau(M, M')$.

This last result can be reconsidered by saying that each rest of a Mackey space is a Mackey space, and that each is metrizable subspace of a Mackey space is Mackey space.

We go to the duality among things and direct sums. Let $\{\langle F, G \rangle : \alpha \in A\}$ we show that a gathering of dualities over K and let $F = \prod_\alpha = F_\alpha$, $G = \oplus_\alpha\ G_\alpha$ the bilinear structure F on $F \times G$, characterized by

$$f(x, y) = \Sigma_\alpha = \langle x_\alpha, y_\alpha \rangle$$

We note that entire is over an and not any more set number of non-zero terms), places F and G in duality; let us denote by $\langle F, G \rangle$ the two Fold system (F,G.F).

As before we will recognize each F_α with the subspaces $F_\alpha \times \{0\}$ of F and each Gα with the subspace $G_\alpha \oplus \{0\}$ of G; regardless, For progressively conspicuous clarity polars concerning $\langle F(\alpha), G_\alpha \rangle$ will be mean by (α∈A)and polars with respect to $\langle F, G \rangle$ by $*$. We Further note that pα is projection $F \to F_\alpha, q_\alpha$ the mixture $G_\alpha \to G(\alpha \in A)$, by then

$$\langle p_\alpha x, y_\alpha \rangle = \langle x, q_\alpha\ y_\alpha \rangle$$

is a character For $x \in F, y_\alpha \in G_\alpha$ and $\alpha \in A$. Subsequently, p_α and q_α are weakly determined with respect to $\langle F, G \rangle$ and $\langle F(\alpha), G_\alpha \rangle$.

In the event that S_α is a gathering of desolately restricted, Floated subsets of $F_\alpha, (\alpha \in A)$, by then it is expeditious that each product $S = \alpha S_\alpha$ is a $\sigma(F, G)$- constrained ,drifted subsets of F; let us demonstrate by $S = \prod \alpha S_\alpha$ the group of all such item sets.

S covers F each S_α covers F(α), (α∈A). Dually , let S be a gathering of weakly constrained , Floated subsets of G_α, (α∈A); by then each set S'=⊕(α∈H) (S'), where H is any restricted subset of A, is encompassed, and σ(F,G)- restricted in G; let us mean by S'=⊕α (S') the group of every single such total. S Covers G if each (S') covers G_α, (α∈A). With this documentation we obtain

Theorem (1.2) : The result of the [S']-topologies is indistinguishable with the S'- topology on F; dually the locally raised direct entire of the S_α-topologies is vague with the S-topology on G.

Proof : In the event that $S' = \oplus_\alpha \in H)\ [S']_\alpha$, where H contains $n \geq 1$ segments, a short estimation shows that

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$$(S')^o \subset \prod_{\alpha \in H} (S_\alpha)^o \times \prod_{\alpha \notin H} F_\alpha \subset n(S')^o,$$

Which demonstrates the primary declaration.

Dually let$S = \prod_\alpha S_\alpha$what′s more, accept each S_α;$\alpha \in A$ to be pitifully closed , convex and orbited. It is obvious that the convex circumnavigated structure $\Gamma\alpha S_\alpha^\circ$ it contained in S° . On the other hand , on the other hand that $y=(y_\alpha) \in S^\circ$, at that point$\sum_\alpha |\langle x_\alpha, y_\alpha \rangle| \leqq 1$ For all $x=x_\alpha \in S$; letting λ_α= sup $\{x_\alpha, y_\alpha : x \in S\}$, it pursue that $\lambda\alpha=0$ except For limitedly numerous $\alpha \in A$ and $\sum_\alpha \lambda_\alpha \leqq 1$ .Now $y_\alpha \in \lambda_\alpha S_\alpha^\circ$; hence $y = \sum_\alpha y_\alpha \in \Gamma_\alpha S_\alpha^\circ$, which demonstrate that $\hat{S}^o = \gamma_\alpha \hat{S}_\alpha^o$. Since the absolutely of sets $\{\Gamma_\alpha S_\alpha^o\}$ structure a 0-neighbourhood base For the locally convex direct entirety of the $\mathfrak{S}_\alpha$- topologies, this topology is indistinguishable with the $\mathfrak{S}$-topology on G.

Theorem (1.3) : Let $\{E_\alpha : \alpha \in A\}$ be a gathering of 1.c.s. Furthermore, let $E = \prod_\alpha E_\alpha$. The two Fold E of E is mathematically with $\oplus_\alpha [E']_\alpha$ and the going with topological characters are significant:

$$\sigma(E, E') = \prod_\alpha \sigma(E_{\alpha,} E'_\alpha).$$

$$\tau(E, E') = \prod_\alpha \tau(E_{\alpha,} E'_\alpha).$$

$$\tau(E', E) = \oplus_\alpha \tau(E'_{\alpha,} E_\alpha).$$

Comment. We have $\sigma(E', E) = \oplus_\alpha \sigma(E'_{\alpha,} E_\alpha)$ if and only if the Family $\{E_\alpha\}$ is constrained confirmation. It is brisk that each $f = (f_\alpha) \in \oplus_\alpha E'_\alpha$ portrays a straight structure $x \to f(x) = \sum_\alpha f_\alpha(x_\alpha)$ on E which is steady, since $f = \sum_\alpha f_{\alpha^\circ} p_\alpha$ (total having only a predetemined number of non-zero terms); obviously, this mapping of $\oplus_\alpha E'_\alpha \alpha$ into E' is coordinated into E' is composed. There remains to show that each $g \in E'$ starts in this plan. There exists a 0-neighbourhood U in E on which g is restricted; U can be anticipated From the structure $\prod_{\alpha \in H} U_\alpha \times \prod_{\alpha \in H} E_\alpha$ For a proper constrained subset $H \subset A$. Show by f_α $(\alpha \in A)$ the restriction of g to E_α; then doubtlessly, $f_\alpha \in E'_\alpha \alpha$ For all α and $f\_\alpha = 0$ if $\alpha \notin H$. Hence For $x \in E$ we acquire

$$g(x) = g\left(\sum_{\alpha \in H} p_\alpha x\right) = \sum_{\alpha \in H} f_\alpha(x_\alpha),$$

Which develop the announcement; E' is therefore isomorphic with the numerical direct entirety $\oplus_\alpha E'_\alpha$ by magnificence of the duality among things and direct aggregates introduced beforehand. It remains to show the topological proposals. IF $\mathfrak{S}'_\alpha$signifies the group of all limited dimensional, limited, subsets of $E'_\alpha (\alpha \in A)$, it is apparent that $\mathfrak{S}\mathfrak{S}' = \oplus_\alpha \mathfrak{S}'_\alpha$s major For the group of all limited dimensional, limited, subsets if $\oplus_\alpha E'_\alpha$; the recommendation Follows.

IF $\mathfrak{S}_\alpha$ means the group of all convex, circumnavigated Feebly smaller subsets of $E_\alpha (\alpha \in A)$, then $\mathfrak{S} = \Pi_\alpha \mathfrak{S}_\alpha$ is a principal sub-Family of the Family C of all convex, circumnavigated, Feebly conservative subsets of E; truth be told, if $C \in C$, then $p_\alpha(C) \in S_\alpha$, since by, $[p]_\alpha$ is Feebly constant on E into $E_\alpha (\alpha \in A)$, and again by Π_α $p_\alpha(C) \in \mathfrak{S}$ again by ethicalness of 1, above, and the Tychonov hypothesis which states that any result of minimized space is reduced, Thus this S-topology on E' is $\tau(E', E)$.

2. IF $\mathfrak{S}'_\alpha$ means the group of all convex, circumnavigated, Feebly minimized subsets of $E'_\alpha (\alpha \in A)$, it does the

trick to demonstrate that $\mathfrak{S}' = \oplus_\alpha \mathfrak{S}'_\alpha$; is a principal system of convex, hovered subsets of E' that are reduced For σ(E',E). In the event that C is such a set, C is limited For σ(E',E) and thus limited For $\tau(E', E)$.

Along these lines, above, C is contained in $\oplus_{\alpha\in H}\bar{p}_\alpha(C)$, where H is a sensible constrained subset of A, and where $\bar{p}_\alpha$ implies the projection o E' on to E'_α. since $\bar{p}_\alpha$ is non-stop For $\sigma(E', E)$ ($\bar{p}_\alpha$is without a doubt, even unsurprising For coarser topology induced on E' by $\prod_\alpha \sigma(E'_\alpha E_\alpha)$) into $(E'_\alpha, \sigma(E'_\alpha E_\alpha))$, it Follows that $\bar{p}_\alpha$ (C) $\in \mathfrak{S}'_\alpha$ it pursues that $p_\alpha(C) \in S_\alpha$ conclusion, since clearly every person From S' is raised, circumnavigated and littler For $\sigma(E^{\wedge'}, E)$.

This completes the affirmation.

Corollary 1 : Let $[\hat{E}']_\alpha(\alpha \in A)$, be a gathering of 1.c.s. moreover, let E be their locally raised direct entirety. E is scientifically isomorphic with $\prod_\alpha E'_\alpha$, what's more, after topological characters are legitimate:

$\tau(E, E') = \oplus_\alpha(E_\alpha E'_\alpha)$.

$\tau(E', E) = \prod_\alpha \tau(E'_\alpha E_\alpha)$.
$\tau(E', E) = \prod_\alpha \sigma(E'_\alpha E_\alpha)$.

Proof : It pursues promptly that the double E' of E can be related to $\prod_\alpha E'_\alpha$, by ethicalness of the accepted duality among items and direct totals; For the remaining asseroom it is adequate to trade E and E'.

Corollary 2 : The things, locally curved direct aggregate, and the inductive outer most compasses of ct gathering of Mackey spaces is a Mackey space.

For things and direct aggregates the result is immediate For inductive purposes of restriction it seeks after then From Corollary 4 of (4.1).

We supply an unequivocal depiction of various gatherings of restricted subsets in the twofold of things and I.c. direct wholes Furthermore that last bit of the proof of is particular, if $\{E_\alpha\}$ is a gathering of 1.c.s. additionally, S is an equicontinuous subset of the twofold $\oplus_\alpha E'_\alpha$of $\prod_\alpha E_\alpha$, by then the projection $\bar{p}_\alpha$ (S) is equicontinuous in $\hat{E}'_\alpha$ For each an, and each constrained entire of eqicontinuous sets is equicontinuous in $\oplus_\alpha \hat{E}_\alpha$. In this manner From 3, it seeks after that $\mathfrak{S}' = \oplus_\alpha \mathfrak{S}'_\alpha$is a key group of equicontinuous sets in $\oplus_\alpha E'_\alpha$if each$\mathfrak{S}'_\alpha$if each $\hat{S}_\alpha$ is such a Family in $\hat{E}'_\alpha$. A comparing result holds if " equicontinuous " is supplanted by " Feebly limited " ; thus, in view of the characterization of equicontinuous sets in the dual of a barreled space.

Corollary 3 : The result of any group of dashed spaces is zoomed. At last we get a portrayal of the double of a space of constant.

Corollary4 : Let E,F be I.c.s. also, mean by L_s (E,F) the space of constant straight maps of E into F under the topology of basic assembly. The correspondence$\sum x_i \otimes y'_i \to F$ defined by

$F(u) = \sum\langle ux_i, y'_i\rangle$ $(u \in L\ (E, F))$,

Is an (arithmetical) isomorphism of $E \otimes F'$onto the dual of $L_s(E.F)$.

Proof : IF $v = \sum x_i \otimes y'_i$,the mapping $v \to F$ is clearly a direct guide of $E \otimes F'$ into L'_s which is likewise biunivocal, since the bilinear structure $(v, u) \to F(u)$places even the subspace $E' \otimes F$ of L (E.F) in isolated duality with $E \otimes F'$. There stays to demonstrate that this mapping is onto L'_s since $L_s(E.F)$is a subspace of the

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item space F^{ε}, every g ∈L's; is the conFinement of a consistent direct structure on F^{ε} thus the structure

$$u \to g(u) = \sum \langle ux_i, y_i' \rangle$$

Subsequently the structure $\{x_i\} \subset E$ and $\{y_i'\} \subset F'$, which cpmpletes the evidence.

We close this segment with an utilization of the Former outcomes to the duality among projective and inductive points of conFinement. Review that a projective Limit $E = \varprojlim g_{\alpha\beta} E_\beta$ is by deFinition ,a subspace of $\prod_\beta E_\beta$to be specific the subspace $\bigcap_{\alpha \leqq \beta} u_{\alpha\beta}^{-1}$ (0),where$u_{\alpha\beta} = p_\alpha - g_{\alpha\beta} \circ p_\beta$ whenever$\alpha \leqq \beta$, As Far as possible E is called diminished if For each α, the projection p_α (E) is thick in E_α . There is no limitation of all inclusive statement in expecting a projective breaking point to be diminished : Letting $F_\alpha = p_\alpha(E)$. (conclusion in Eα) and indicating $u_{\alpha\beta}$ restriction of $u_{\alpha\beta}$, to $\prod_\beta F_\beta$ is identical with the subspace $\bigcap_{\alpha \leqq \beta} u_{\alpha\beta}^{-1}$ (0), of $\prod_\beta F_\beta$.

Signifying by $h_{\beta\alpha}$ boa the adjoint of $g_{\alpha\beta}$ concerning the dualities$\langle E_\alpha E_\alpha' \rangle$ and$\langle E_\beta E_\beta' \rangle (\alpha \leqq \beta)$; it pursues (since$g_{\alpha\beta}$is pitifully ceaseless) that $h_{\beta\alpha}$ is persistent For the Frail and Mackey topologies, individually, on $E_\beta'$and E_α' . Moreover, $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma} (\alpha \leqq \beta \leqq \gamma)$ implies $h_{\gamma\alpha} = h_{\gamma\beta} \circ h_{\beta\alpha}$ boa.

Theorem (1.4) : IF $E = \lim_{\leftarrow} g_{\alpha\beta} E_\beta$is projective point of conFinement of 1.c.s., at that point the double E', under its Mackey topology $\tau(E', E)$, can be related to the inductive Furthest reaches of the Family $\{(E_\alpha' v(E_\alpha', E_\alpha))\}$ as For the adjoint mappings $h_{\beta\alpha}$of $= g_{\alpha\beta}$.

Proof : Let $F = \oplus_\alpha E_\alpha'$, where each E_α', is invested with$\tau(E_\alpha', E_\alpha)$. By definition$\varinjlim h_{\beta\alpha} E_\alpha'$is the remainder space F/H_O(if H_O is closed in F), where H_O is the space of F produced by the reaches $v_{\beta\alpha}(E_\alpha')$, where $V_{\beta\alpha} = q_\alpha\ q_\beta\ h_{\beta\alpha}\ (\alpha \leqq \beta)$.

We demonstrate that H_O is the subspace of F symmetrical to E as For the duality$\langle \prod_\alpha E_{\alpha,} F \rangle$. E° is the Feebly closed, convex structure of $\bigcup_{\alpha \leqq \beta} [u_{\alpha\beta}^{-1}(0)\}^\circ$, which in perspective on the result equivalent to the pitifully closed, convex structure $\bigcup_{\alpha \leqq \beta} v_{\beta\alpha}(F)$;this suggests $H_O \subset E^\circ$. Conversely,let $y = (y_\alpha)$be a component of E° , let H be the limited arrangement of records such that α∈H if and just if $y_\alpha \neq 0$ also, pick a record β to such an extent that α≦β For all α∈H ; at long last given *x* a chance to be any component of E. At that point we have

$$\langle x, y \rangle = \sum_{\alpha \in H} \langle x_\alpha, y_\alpha \rangle = \sum_{\alpha \in H} \langle g_{\alpha\beta}, x_\beta, y_\alpha \rangle = \sum_{\alpha \in H} \langle x_\beta, h_{\beta\alpha}, y_\alpha \rangle$$

$$= \langle x_\beta, \sum_{\alpha \in H} h_{\beta\alpha}, y_\alpha \rangle = 0;$$

since by supposition x_β goes through a thick subspace of E_β as x goes through E, the previous connection suggests that$\sum_{\alpha \in H} h_{\beta\alpha}, y_\alpha = 0$, hence $y = \sum_{\alpha \in H} (q_\alpha - q_\beta \circ h_{\beta\alpha}) y_\alpha \in H_O$.

Thus H_O is pitifully closed in F, thus closed For $\tau(F, \prod_\alpha E_\alpha)$, which by (4.3) is the topology $\oplus_\alpha \tau(E_\alpha', E_\alpha)$;hence as Far as possible $\varinjlim h_{\beta\alpha} E_\alpha'$ of the Mackcy duals$(E_\alpha', \tau(E_\alpha', E_\alpha))$exists and topology is the topology$\tau(F/H_O, E)$, which demonstrates it to be isomorphic with the Mackey double $(E', \tau(E', E))$ of E. With the guide of, we presently eFFectively get the accompanying double outcome For inductive breaking points:

Theorem(1.5) : Let $E = \varinjlim h_{\beta\alpha} E_\alpha$be inductive Farthest point of l.c.s. The powerless double of E is isomorphic with the projective Furthest reaches of the Feeble duals$(E_\alpha', \sigma(E_\alpha', E_\alpha))$as For the adjoint maps$g_{\alpha\beta}$ of $h_{\beta\alpha} (\alpha \leqq \beta)$.

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Comment - On the other hand that the duals $\hat{E}'_\alpha$ are supplied with their individual Mackey topologies, at that point it pursues From (2.3), Corollary I, and (2.1), Corollary 3, that the projective Furthest reaches of these duals, arithmetically ideati6ed with E', carries a topology $\mathfrak{T}$ which is reliable with $\langle E, E' \rangle$. Thus if $\mathfrak{T}$is known to be the Mackey topology (specifically, if $\mathfrak{T}$ is metrizable), at that point the Mackey double of E can be related to the projective Furthest reaches of the Mackey duals E'_α .

CONCLUSION

This paper strenuously examines the rich scenery of locally convex topological vector spaces with special attention to duality properties in the context of projective and inductive topologies. Through the careful analysis of how duality occurs throughout the structures, this article effectively spans classical basic ideas and analytical advancements.

Our investigation demonstrates a profound understanding of special classes of locally convex spaces, including the behavior of projective limits, inductive limits, convex bornological spaces, and (DF)-spaces under various duality contexts. The established results, such as the characterization of how topologies are induced and quotiented under duality pairings, provide a clearer picture of the intricate relationships between a space and its dual within these limit constructions. Specifically, theorems regarding the duality of products and direct sums of locally convex spaces, and their consequences with respect to Mackey spaces, further the more general theory of topological tensor products and sum topologies in infinite dimensions.

One of the greatest contributions of this paper is to establish the key conditions of consistency, e.g., the consistency of certain topologies with twofold systems, that form the basis of understanding the continuity and convergence properties in these abstract spaces. Additionally, the explicit description of different families of bounded subsets in the dual of products and direct sums further enriches the toolkit of researchers in locally convex spaces. The discussion is a culmination of explaining the duality of projective and inductive limits, showing how the dual of a projective limit is equivalent to an inductive limit of duals and vice-versa under certain topological considerations. These results are not so much theoretical exercises as they are providing vital machinery to address problems in functional analysis, operator theory, and the study of function spaces, where the interaction of topological structures and their duals is critical. This article therefore solidifies the analytical groundwork for coming advances in the theory of locally convex spaces and their applications.

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