A Proof of the Frattini Argument about Normal Series

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DOI: https://doi.org/10.51244/IJRSI.2025.120700217

Received: 16 July 2025; Accepted: 24 July 2025; Published: 22 August 2025

ABSTRACT

The Frattini Argument has notoriously many different stories and versions that a casual researcher may get confused about what this important statement is really all about. We are not going to discuss the different versions here; we want to supply a proof of one version that does not yet have a proof today. Every version of the Frattini Argument is attributed to the Italian mathematician Giovanni Frattini (1852-1925). The specific version of Frattini Argument that is our focus is concerned with the existence of a **Series of Normal Subgroups** below a finite group. Important as this theorem is, it has however not been proved by anyone, including Frattini himself. Although it has been generally accepted to be true—and used in analysis—no one has actually supplied a comprehensive proof of this theorem.

Keywords: Group. Subgroup. Normal Subroup. Series of Groups. Maximal Subgroup. Mathematics Subjects Classification (MSC) 2020: 20D05, 20D25, 20D30, 20E28

INTRODUCTION

Definition 1.1 *A group* $(G, *)$ *is a set G equipped with a binary operation* $*$ *such that*

1. *The associative law holds; namely, for every* $x, y, z \in G$, *we have*

$x * (y * z) = (x * y) * z.$

2. *There is an element* $e \in G$, *called the identity element, with* $e * x = x = x * e$, *for all* $x \in G$.

3. *Every* $x \in G$ *has an inverse element* x^{-1} *such that* $x * x^{-1} = x^{-1} * x = e$.

Definition 1.2 *A group* $(G, *)$ *is abelian if* $x * y = y * x$ *for all* $x, y \in G$.

Definition 1.3 *A subset H of a group G is called a subgroup of G, written*

$H \leq G$, *if* $(H, *)$ *satisfies the properties of a group as outlined in definition 1.1*

Remark

We often omit the explicit writing of the binary operation when describing group operations; and so we write the operations *multiplicatively*. For example, instead of writing $x * y$ we just write xy.

Definition 1.4 *If* $H \leq G$ *and for all* $h \in H$, *and* $g \in G$ *we have* $ghg^{-1} \in H$, *or* $H^g = H$, *or* $gHg^{-1} = H$, *we say that H is a normal subgroup of G. We denote this by* $H \triangleleft G$.

Definition 1.5 *Let G be a group and let* $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$. *Then* $Z(G)$ *is called the center of the group G.*

Theorem 1.1 *Any subgroup of the center* $Z(G)$ *of a group G is a normal subgroup of G.*

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Proof:

We show that $[Z(G)]^g = Z(G)$ for all $g \in G$. Let $z \in Z(G)$ and $g \in G$. Then taking conjugate $z^g = g^{-1}zg = g^{-1}gz = z \in Z(G)$, since elements of $Z(G)$ commute with every other element of G. This implies that $[Z(G)]^k = \{z^g : z \in Z(G)\} = \{g^{-1}zg : z \in Z(G)\} = \{z : z \in Z(G)\} = Z(G)$.

That is, $[Z(G)]^k = [Z(G)]$. That is $[Z(G)] \triangleleft G$. In a similar way, it is easy to show that every subgroup of $Z(G)$ is a normal subgroup of G.

■

Lemma 1.1 (Correspondence Lemma) *Let G be a group and let K be a normal subgroup of G. Let* $\pi: G \longrightarrow G/K$ *be the natural map such that* $\pi(g) = gK$.

Let $\Psi = \{S \subset G : K \subset S\}$ *and let* $\Phi = \{S/K : S \in \Psi\}$. *Define* $f: \Psi \to \Phi$ *by*

$f(S) = S/K$. *Then f is a bijection.*

Proof:

First we see that if $K \subset S \subset G$, then the set $\{gK : g \in S\} = S/K = \pi(S)$ is well defined. If $f(S_1) = f(S_2)$, then it follows that $S_1/K = S_2/K$. This implies that $S_1 = S_2$. Hence f is one-to-one (injective). Also for any $S/K \in \Phi = \{S/K : S \in \Psi\}$, there exists $S \in \Psi$ such that $f(S) = S/K$. Therefore f is surjective and so a bijection.

Remark

For a proof of the next theorem, see theorem 2.103 on page 104 of [1] .

Theorem 1.2 *If p is a prime and G is a p-group, then* $Z(G) \neq \{e\}$, *that is the center of G is a non-trivial subgroup of G.*

Proposition 1.1 *If p is a prime number and G is a group of order* $|G| = p^n$, *for a non-negative integer n, then G has a normal subgroup of order* p^k *for every* $k \leq n$.

Proof:

We prove the result by induction on n. If $n = 0$ then $|G| = p^0 = 1$. This implies that G is the trivial group $G = \{e\}$, a normal subgroup of itself. If $n \geq 1$, then G is a p-group and by theorem 1.2, the center of G is nontrivial, $Z(G) \neq \{e\}$. By theorem 1.1 any subgroup of the center $Z(G)$ of a group G is a normal subgroup of G. So, let $T \leq Z(G)$ be a subgroup of $Z(G)$ whose order is $|T| = p$. Then T is a normal subgroup of G. By Lagrange's theorem below, the order of T divides the order of G, and since $|G| = p^n$, it follows that

$$|G/T| = |G|/|T| = p^n/p = p^{n-1}.$$

If $k \leq n$ then $k-1 \leq n-1$ and $p^{k-1} \leq p^{n-1} = |G/T|$. Since T is a normal subgroup of G, the quotient group G/T is a normal subgroup of G. (Recall that the quotient of a group by a normal subgroup is a normal subgroup.) So G/T is a normal subgroup of G of order p^{n-1}. Since we can observe that this happens as $p^{k-1} \leq p^{n-1} \leq p^n$ by inductive reasoning, the subgroup G/T of G order p^{n-1} has a normal subgroup (say) K of order p^{k-1}. It follows from the correspondence lemma above that there exists a subgroup H of G such that $T \subset H$ and

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XII Issue VII July 2025

$$H/_T = K.$$

Also the equation $H/_T = K$ and the fact that K is a normal subgroup of $G/_T$ implies that $H/_T$ is a normal subgroup of $G/_T$; since $H/_T = K \lhd G/_T$. That is

$$H/_T \lhd G/_T.$$

Hence H is a normal subgroup of G. But

$$H/_T = K \implies |H/_T| = |K| = p^{k-1}.$$

As we now recall that $|T| = p$, it follows that

$$|H/_T| = |H|/_{|T|} = |H|/_p = |K| = p^{k-1},$$

from which we finally have that

$$|H| = p^{k-1}.p = p^k.$$

Therefore G has a normal subgroup H of order p^k for every $k \leq n$.

Remark

The above proof of proposition 1.1 relied more on the proof of proposition 2.106 given on page 106 of Joseph J. Rotman's book. See [1].

Theorem 1.3 (Lagrange) *If H is a subgroup of a finite group G, then the order |H| of H divides the order |G| of G.*

Proof:

Let $\{a_1H_1, a_2H_2, \cdots, a_kH_k\}$ be the family of all the distinct cosets of H in G. Then since this family partitions G, G is the union of this pairwise disjoint sets; namely

$$G = a_1H_1 \cup a_2H_2 \cup \cdots \cup a_kH_k,$$

because each $g \in G$ lies in the coset gH, and $gH = a_iH$ for some $1 \leq i \leq k$. Therefore the order of G is

$$|G| = |a_1H_1| + |a_2H_2| + \cdots + |a_kH_k|.$$

But the order $|a_iH_i|$ of each coset of H in G equals the order of H; that is, $||a_iH_i = |H|$. (This is because $aH = \{ah : h \in H\}$ for each $a \in G$.) Therefore we have

$$|G| = |a_1H_1| + |a_2H_2| + \cdots + |a_kH_k| = k|H|.$$

Since k is a positive integer, the last equation means that $|H|$ divides $|G|$'

The following theorem will be used in our proof of the Frattini argument.

Theorem 1.4 *Let G be a finite p-group (in the sense that the order of G is a power of a prime number p).*

1. *If $H < G$, then $H < N_G(H)$. This is called the normalizer condition.*
2. *If M is a maximal subgroup of G, then M is normal in G and $[G : M] = p$.*

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Proof:

First we recall that $N_G(H) = \{g \in G: H^g = H\}$. Hence (as $H^h = H$ for all $h \in H$) it follows that $H \leq N_G(H)$. We now have to show that $H < N_G(H)$.

We proceed with induction on the order $|G|$ of G. Suppose $|G| = p$, a prime number. Then since the order $|H|$ of H must divide the order of G, by Lagrange's theorem, and since $H < G$, it follows that $H = \{e\}$, the trivial subgroup of G consisting of only the identity element of G. But $H = \{e\} \Longrightarrow H^g = \{e\}^g = g^{-1}\{e\}g = \{e\} = H$, for all $g \in G$. This implies that $N_G(H) = G$. Therefore $H < N_G(H)$, as $G = N_G(H)$ and from hypothesis $H < G$.

Now suppose $|G| = p^\alpha > p$. Then if the center $Z(G)$ of G is not a subset of H, it must contain an element not in H. That is, $Z(G)$ not a subset of H implies that there exists x in $Z(G)$ such that $x \notin H$. (Now we recall that $H.Z(G) = \{hz : h \in H, z \in Z(G)\} = \{hz: h \in H, zg = gz$ for all $g \in G\}$.) In particular, as $x \in Z(G)$, $xh = hx \in H.Z(G)$ and $xh \notin H$, since $x \notin H$.

Hence $H < H.Z(G)$. . . . . (1)

We observe that $H.Z(G) \leq G$. We also observe that if $g = hz \in H.Z(G)$, then $H^g = g^{-1}Hg = (hz)^{-1}H(hz) = (z^{-1}h^{-1}Hhz) = h^{-1}Hh$, since $z \in Z(G) \Longrightarrow z^{-1}h^{-1}Hhz = h^{-1}H(z^{-1}z)h = h^{-1}Hh = H$. That is, $H^g = H$ if $g \in H.Z(G)$.

This implies that $H.Z(G) \leq N_G(H)$. . . . . (2)

Therefore since we have seen from (1) that $H < H.Z(G)$, it follows that $H < H.Z(G) \leq N_G(H)$. Thus $H < N_G(H)$, as desired.

Now suppose that $Z(G)$ is a subset of H, so that $Z(G) \leq H$. Then since $Z(G)$ is a normal subgroup of G, it is also a normal subgroup of H. Since G is a p-group, $Z(G) \neq \{e\}$ = the trivial subgroup of G. Also since $H < G$ and $Z(G) \neq \{e\}$,

$$H \neq \frac{H}{Z(G)} \text{ and } \frac{H}{Z(G)} < \frac{G}{Z(G)}.$$

The last relation implies, by analogy from what we have just proved in the last paragraph, that the normalizer of $\frac{H}{Z(G)}$ in $\frac{G}{Z(G)}$ is a proper *supergroup* of $\frac{H}{Z(G)}$; conversely that $\frac{H}{Z(G)}$ is a proper subgroup of the normalizer of $\frac{H}{Z(G)}$ in $\frac{G}{Z(G)}$. That is

$$\frac{H}{Z(G)} < N_{\frac{G}{Z(G)}}\left(\frac{H}{Z(G)}\right) = \frac{N_G(H)}{Z(G)} \text{ . . . . (3)}$$

To see the equality in (3), we first let

$$\Psi = N_{\frac{G}{Z(G)}}\left(\frac{H}{Z(G)}\right) = \{y \in \frac{G}{Z(G)} : [\frac{H}{Z(G)}]^y = \frac{H}{Z(G)}\}.$$

So, if $y \in \Psi$ then $\left[\frac{H}{Z(G)}\right]^y = \frac{H}{Z(G)}$.

This implies that $y^{-1}\left[\frac{H}{Z(G)}\right]y = \frac{H}{Z(G)}$ or that $\left[\frac{H}{Z(G)}\right]y = y\left[\frac{H}{Z(G)}\right]$.

Therefore $([Z(G)]h)y = y([Z(G)]h)$, for all $h \in H$.

$\Longrightarrow [Z(G)]hy = yh[Z(G)]$, for all $h \in H$, ... (4)

because $[Z(G)]h = h[Z(G)]$, for all h. This property of $Z(G)$ when applied to

(4) implies that $hy[Z(G)] = yh[Z(G)]$, for all $h \in H$. . . . . (5)

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Therefore $hy = yh$, for all $h \in H$ and $y \in \Psi$. . . . . (6)

But $y \in \Psi$ implies that $y \in \frac{G}{Z(G)}$. This implies that y is of the form

$$y = [Z(G)]g, \text{ for some } g \in G \text{ . . . . (7)}$$

Putting (7) in (6), for y, gives

$$h([Z(G)]g) = ([Z(G)]g)h$$

This implies that $[Z(G)]hg = [Z(G)]gh$, or that $hg = gh$, for some $g \in G$ and all $h \in H$. The last statement means that g is an element of the normalizer of H in G; that is, $g \in N_G(H)$. Therefore if $y \in \Psi$, then $y = [Z(G)]g$, where $g \in N_G(H) = \{g \in G : H^g = H\}$. But clearly the relation

$$y = [Z(G)]g \text{ and } g \in N_G(H)$$

simply means that y is an element of the quotient group of $Z(G)$ in $N_G(H)$. In other words

$y \in (\frac{N_G(H)}{Z(G)})$. . . . . (8)

It then becomes clear from (8) and the definition of Ψ, in (3), that

$N_{\frac{G}{Z(G)}}(\frac{H}{Z(G)}) \leq \frac{N_G(H)}{Z(G)}$ . . . . (9)

Now suppose that $y \in \frac{N_G(H)}{Z(G)}$. Then it follows that $y = [Z(G)]g$, for some $g \in N_G(H)$.

But $g \in N_G(H)$ implies that $gh = hg$, for all $h \in H$. So

$y \in \frac{N_G(H)}{Z(G)} \Longrightarrow y = [Z(G)]g$ . . . . (10)

where $g \in G$ and $gh = hg$ for all $h \in H$. To show that $y \in N_{\frac{G}{Z(G)}}\left(\frac{H}{Z(G)}\right)$, we need to prove that $y \in \frac{G}{Z(G)}$ and that $\left[\frac{H}{Z(G)}\right]^y = \frac{H}{Z(G)}$. Using the fact that the right coset expressing y in (10) is an equivalence class, we write y as $y = zg$, where $z \in Z(G)$. Then we have

$[\frac{H}{Z(G)}]^y = g^{-1}z^{-1}(\frac{H}{Z(G)})zg = g^{-1}(\frac{H}{Z(G)})g = \frac{H}{Z(G)}$, . . . (11)

because first $\left[\frac{H}{Z(G)}\right]^y = y^{-1}\left[\frac{H}{Z(G)}\right]y = (zg)^{-1}\left[\frac{H}{Z(G)}\right](zg) = g^{-1}z^{-1}\left[\frac{H}{Z(G)}\right]zg.$

The last equation in (11) proves that

$$\left[\frac{H}{Z(G)}\right]^y = \frac{H}{Z(G)}\text{. . . . . (12)}$$

From (10), it is also true that

$y \in \frac{G}{Z(G)}$. . . . . (13)

Thus (13) and (12) together prove that $y \in \frac{G}{Z(G)}$ and that $\left[\frac{H}{Z(G)}\right]^y = \frac{H}{Z(G)}$. In other

words,

$y \in N_{\frac{G}{Z(G)}}(\frac{H}{Z(G)})$. . . . . (14).

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Since $y \in \frac{N_G(H)}{Z(G)}$ was arbitrarily chosen after line (9), (14) implies that

$$\frac{N_G(H)}{Z(G)} \leq N_{\frac{G}{Z(G)}}\left(\frac{H}{Z(G)}\right). \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (15)$$

Finally the inequalities in (9) and (15) prove the equality in (3).

And from (3) it follows that $\frac{H}{Z(G)} < \frac{N_G(H)}{Z(G)}$, from which it also follows that

$H < N_G(H)$, marking the end of the proof of part 1 of this theorem. Next we prove part 2.[1]

Since M is a maximal subgroup of G, it is a proper subgroup of G. By part

1 of the theorem just proved, $M < G \Longrightarrow M < N_G(M)$. Therefore we have

$$M < N_G(M) \leq G. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (16)$$

By the maximality of M, the inequalities in (16) imply that $N_G(M) = G$. That is, the normalizer of M in G equals G; or $N_G(M) = \{g \in G : M^g = M\} = G$. This result means (from definition) that M is a normal subgroup of G.

Now, since M is a Sylow p-subgroup of G (i.e. a maximal p-subgroup of G), and since G is a finite p-group, if $|G| = p^m$, then $|M| = p^{m-1}$ (otherwise M would not be maximal). This shows that

$$|G : M| \equiv \frac{|G|}{|M|} = \frac{p^m}{p^{m-1}} = p^{m-m+1} = p.$$

That is, $[G\colon M] = |G\colon M| = \left|\frac{G}{M}\right| = \frac{|G|}{|M|} = \frac{p^m}{p^{m-1}} = p^{m-m+1} = p$, as desired. The entire proof is complete.

Remark

We now state and prove the Frattini Argument[2] about normal series.

1 Main Result: Proof of the Frattini Argument About Normal Series

Theorem 2.1 (Frattini) *Let G be a finite group and H a subgroup of G. If the order |H| of H is a power of a prime number p (i.e. H is a p-subgroup of G), then G has a series of normal subgroups. In notations, if $H \leq G$, and G is finite, and $H = p^n$ for some p a prime number and $n \in N = \{0,1,2,3,\cdots\}$, then there exist $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_i = H \leq G$ such that H_{i-1} is a normal subgroup of H_i, for all i, that is $\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_i = H \triangleleft G$ and $[H_{i+1} : H_i] = p$ for all i.*

Proof

First, we note that G is not necessarily a p-group. If G is a p-group, then the result is a corollary of theorem 1.4 (2); for then H is either a Sylow p-subgroup of G (i.e. a maximal p-subgroup of G) or a Sylow p-subgroup of another maximal p-subgroup of G. Since Sylow p-subgroups are normal, either of these cases generates the required series of normal subgroups below G. Then finally, for any divisor of $|H| = p^i$ there exists (from proposition 1.1) a normal subgroup of H. Such subgroups form a series of normal subgroups below H; and they are a series of normal subgroups below G.

[1] For a much shorter proof of the theorem, see theorem 4.6 on page 75 of *An Introduction to the Theory of Groups*, Fourth Edition, by Joseph J. Rothman [2]. There the entire proof is done in only 10 lines. Our purpose here is to give a more detailed and somewhat clearer proof of the theorem, before applying it in our proof of the Frattini Argument for normal series.

[2] As said in the Abstract many things are called Frattini argument by many authors. See for instance page 81 of [2] for one such. It is not about series at all.

Now suppose that G is not a p-group. Then, since by the Lagrange theorem, the order of H divides the order of G, and since H is a p-group, the order of G is of the form $|G| = p^{\alpha}m$ where $|H| = p^i$, $i \leq \alpha \in N$ = the set of positive integers, and p does not divide m. If H is a Sylow p-subgroup of G (i.e. a maximal p-subgroup of G), then it is a normal subgroup of G, by theorem 1.4 (2), and since (by proposition 1.1) all other powers of p less than $p^i = |H|$ have pairwise comparable normal subgroups below H (and indeed below G) corresponding to them, the series required is generated. For example, p^{i-1} is a power of p less than p^i and there exists a normal subgroup H_1 of H such that $|H_1| = p^{i-1}$; that is $H_1 \triangleleft H \triangleleft G$. Similarly $p^{i-2} < p^{i-1}$ and H_2 with order $|H_2| = p^{i-2}$ is a normal subgroup of H_1, below H and G. We have the normal series as $H_2 \triangleleft H_1 \triangleleft H \triangleleft G$. Since G is finite, the process will have an end, and we shall finally have the desired series as

$$\{e\} = H_0 < H_1 \leq H_2 \leq \cdots \leq H_{i-2} \leq H_{i-1} \leq H_i = H \leq G \ . \ . \ . \ (*)$$

where $|H_1| = p$, $|H_2| = p^2$, $\cdots$, $|H_i| = p^i$, and $\{e\}$ is the trivial (normal) subgroup which consists of only the identity element e of the group G. It is easy to see from the relations in $(*)$ that

$$[H_i : H_{i-1}] = \frac{|H|}{|H_{i-1}|} = p$$

for all $i \geq 0$. This means that $[H_{i+1} : H_i] = p$ for all i.

SUMMARY AND CONCLUSION

1. It is clear from literature that several things are called **the** Frattini Argument by several people. Some of the references—and search engines—can be consulted to verify this.

2. There is one, and only one Frattini Argument and it has not been proved; and it is about normal series. This is the reason why it is referred to as '**the'** Frattini ***Argument***, meaning that it is unique and its proof or disproof has not been provided. This is similar to what was known as Euler's Conjecture (made in 1769) which was disproved in 1987 (218 years later) by Noam Elkies. See [3]

3. Any literature—before our work—containing the Frattini Argument **about normal series** will not contain its proof. This is what motivated us to undertake this lofty task.

REFERENCES

1. Joseph J. Rotman ; Advanced Modern Algebra, Second Printing (2003); Prentice Hall
2. Joseph J. Rotman; An Introduction to the Theory of Groups; Fourth Edition; Springer, Graduate Texts in Mathematics (148)
3. Eric Lehman, F. Thomson Leighton and Albert R. Meyer; Mathematics for Computer Science; Massachusetts Institute of Technology; Revised on Wednesday, 6th June, 2018.
4. Serge Lang; Algebra; Third Edition; Addison Wesley Publishing Company
5. Titchmarsh E.C.; Theory of Functions; Second Edition, Oxford University Press, Oxford (1939).
6. George D. Mostow, etal; Fundamental Structures of Algebra;
7. MacGraw Hill Book Company
8. Michio Suzuki; Group Theory; Springer-Verlag (1980).
9. Angus E. Taylor and David C. Lay; An Introduction to Functional Analysis; Second Edition, John Wiley and Sons, New York (1980).
10. Edwards R.E.; Functional Analysis: Theory and Applications; Dover Publications Inc., New York (1995).
11. H.L. Royden; Real Analysis; Third Edition, Prentice-Hall of India Private Limited, New Delhi (2005).
12. Rudin W.; Functional Analysis; McGraw-Hill, New York
13. (1973).