Topological Properties of Quasilinear Spaces and Set-Valued Maps

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ABSTRACT

This paper presents a comprehensive study on the topological properties within the framework of quasilinear spaces, particularly focusing on their application to set-valued maps. We generalize the established concept of normed quasilinear spaces, primarily as introduced by Aseev in 1986, to the more encompassing notion of topological quasilinear spaces1. Our investigation demonstrates that if S is an arbitrary topological space and Z is a metrizable topological vector space, then the space of minimal semi-continuous compact-valued maps, denoted as M(S,Y), can be mapped into a linear space. This significantly extends previous results where S was restricted to being a Baire space.

A key contribution of this work is the detailed analysis of the topological vector space $M_s(S, Z)$, defined by the topology of strong uniform convergence on compact subsets of S. We establish several fundamental properties of this space, including its completeness when S is a locally compact Hausdorff space and Z is complete. Furthermore, we prove that $M_s(S, Z)$ is metrizable if and only if S is hemicompact. The paper also undertakes a comparative study of the topologies τ_s and τ_k, demonstrating that τ_s is finer than τ_k, with equality holding when S is locally compact. Through rigorous mathematical illustrations and examples, we show that the principle of localization, typically valid for topological vector spaces, may not hold universally for all topologically quasilinear spaces, particularly in the context of singular elements. These findings collectively contribute fresh insights into the topology of quasilinear spaces, multivalued mappings, and set-valued analysis.

Keywords: Quasilinear Spaces, Set-Valued Maps, Topological Properties, Topological Vector Spaces, Metrizability, Completeness, Uniform Convergence, Hausdorff Space, Hemicompact Space, Minkowski Operations, Set-Valued Analysis, Aseev.

INTRODUCTION

The concept of Quasilinear Spaces introduced by Aseev encompassed both the traditional linear spaces & subset of non linear spaces and its multilevel mappings. This he applies to the quasilinear space for its linear functional analysis by introducing the concept of quasilinear operators & functional. From here, he proceeds further to exhibit some outcomes which in linear functional analysis context happens to be the quasilinear equivalent of the functional definition & theorem whereas in Banach Spaces context they were differential calculus. Aseev's this path breaking work has motivated many, and this paper, to present new results on fuzzy quasilinear spaces, multileveled mappings & set-valued analysis.

The most significant example of a quasilinear space is $K_c(E)$ when $K_c(E)$ is a collection consisting of all convex compact subset of E, and where E is a normed space. The study of this class includes convex and interval analysis since intervals turns out to be an ideal tools in the context of solving the global optimization problems as well as in the context of being an addition to the conventional techniques.

Intervals being a set with infinite number, possesses infinite amount of information i.e. worldwide information. Further, the set differential equations theory also requires the study of $K_C(E)$. There are various different methods including Markow method for introducing and for dealing with the quasilinear spaces. However, this paper but feels Aseev's best suited to provide the foundation and tools. Also, the Aseev's method provides higher dimension of set-valued algebra & analysis through ordering relation. Here in this paper the issues related to the problem of uncertainty and to sensitivity necessarily belongs to set-valued analysis. The origin of set-valued analysis traced back to 19th century, developed primarily by Cauchy, Riemann and Weierstrass, has gathered much interest these days. Aseev's early research in this aspect is also worthy of mention here. The main conclusions of this paper show that from S, an arbitrary topological space, the collection of minimal semi-continuous compact value maps into Z a metrizable topological vector space with Z being a vector space, this generalize the former results where S necessarily were Baire Space. The definitions of the algebraic operation, there, depended basically on the fact that any minimal upper semi-continuous compact valued map from S, a Baire space, to Y where Y is a metric space is point-valued on a dense G-δ set. Since this does not hold for generalized topological space S, new techniques have been considered in this paper. The metric characterization of the quasi-minimal upper semi-continuous compact valued maps being the key.

In this paper, until and unless defined otherwise, S and Y continue to denote general topological spaces and Z continues to be a topological vector space over a field K, wherein K is either the field of real numbers, R or C a field of complex numbers. For each s ∈ S, Vx still denotes the collection of an open neighborhoods of s in S. whereas the closure of A ⊆ S still remain to be denoted by A, such that Int(A) is the interior of A. In case Y is a metric space, it is denoted by B(y, ϵ). The open ball with center at y ∈ Y of radius ϵ > 0, where B(y, ϵ) denotes closure of B(y, ϵ).

Wherever it does so necessarily highlight the specific metric d on Y, the ball with center at y ∈ Y of radius ϵ is represented by Bd(y, ϵ). The collection consisting of all the subsets of Y is denoted here as P (Y), whereas C(Y) denotes it as consisting of all closed and non-empty subsets of Y. The collections consisting of all nonempty compact subsets of Y is denoted here as K(Y).

There following the establishment of normed quasilinear spaces and the bounded quasilinear operators in the already established norms, and by introducing some new results, this paper is able to some significant contributions towards the enhancement of the quasilinear functional analysis.

Algebraic Operations and Linear Structures of Set-Valued Maps:

For the sum of subsets A and B of a topological vector space Z, Minkowski defines as follow
$A \oplus B = \{a + b : a \in A, b \in B\}$,
and the Minkowski product of $A \subseteq Z$ and a scalar α is
$\alpha \odot A = \{\alpha a : a \in A\}$. (1)
owing to the continuity of the algebraic operations on Z, K(Z) is closed under the Minkowski operations. That is,

$\oplus$: K(Z) × K(Z) → K(Z)	(2)
And	
$\odot$: K × K(Z) → K(Z).	(3)

Here, though. K(Z) isn't a vector space with respect to (2) and (3). But naturally, although {0} ∈ K(Z) is an identity for (2) and the associativity and commutativity axioms are true, a general member in K(Z) doesn't have an additive inverse. In fact, A ∈ K(Z) have additive inverse if and only if A is a singleton. Furthermore, scalar multiplication (3) isn't distributive over addition in K.

Using operations (2) and (3) in a point-wise manner to employ mappings f, g : S → Z and α ∈ K, one has:

$f \oplus g : S \ni s \to f(s) \oplus g(s) \in$ K(Z),	(4)
And	
$\alpha \odot f : S \ni s \to \alpha \odot f(s) \in$ K(Z)	(5)

are usco. Also, since Minkowski operations do not meet the axioms of linear space on K(Z), it is clear that the set of all usco maps form S into Z is not a linear space under (4) and (5). By proceeding with this, we can observe that an usco map $f : S \rightrightarrows Z$ have an additive inverse if and only if f is point-valued at each s ∈ S. This being the case it follows that f is a continuous function from S into Z. Thus, neither M(S, Z) nor Q(S, Z) is, in general, a linear space according to (4) and (5). Actually, M(S, Z) is not even closed under Minkowski addition

Example Consider the musco maps $f, g : R \rightrightarrows R$ be defined as

$$f(x) = \begin{cases} 0 & if\ x < 0 \\ (1,0) & if\ x = 0 \\ 1 & if\ x > 0 \end{cases} \tag{6}$$

And,

$$g(x) = \begin{cases} 0 & if\ x < 0 \\ (-1,0) & if\ x = 0 \\ -1 & if\ x > 0 \end{cases} \tag{7}$$

Therefore, Minkiwski sum of f ang g will be

$$f \oplus g(x) = \begin{cases} 0 & if\ x \neq 0 \\ (-1,1,0) & if\ x = 0 \end{cases} \tag{9}$$

Now, it is clear that $f \oplus g$ is usco, but not minimal.

M (S, Z) As A Linear Space:

Assuming Z to be metrizable. M*(S, Z)* is not a linear space with and is also not closed under Minkowski addition and hence also not even a quasilinear space. However, the pointwise addition in (4) can be used to define the sum of two musco maps in the following way: for $f, g \in$ M*(S, Z)*, define the *sum* $f + g$ of f and g as $f + g = \langle f \oplus g \rangle$,

where $\langle\cdot\rangle$: Q*(S, Z)* → M*(S, Z)* is the map defined and $f \oplus g \in$ Q*(S, Z)* since f and g, being musco, a quasi-minimal usco. Therefore $f + g$ defined for all $f, g \in$ M*(S, Z)*. Easily it can be seen that M*(S, Z)* is closed under Minkowski scalar multiplication (5). Therefore we defined the scalar product of $\alpha \in$ K and $f \in$ M*(S, Z)* to be the Minkowski product of α and f.

That means,

$$\alpha f = \alpha \odot f. \tag{10}$$

Theorem 1: M(S,Z) is a linear space

Proof: From the above operations. Commutativity of the addition, the existence of the additive identity, distributivity of the scalar multiplication over addition in M(S, Z), the compatibility of the scalar multiplication with field multiplication and also the identity element for scalar multiplication is the multiplicative identity in K all follow immediately, We only checked the rest of the axioms of the linear space.

To find that the addition is associative, assume $f, g, h \in$ M*(S, Z)*. By now, it is clear that

$f + (g + h) \subseteq f \oplus (g + h) \subseteq f \oplus (g \oplus h)$ and
$(f + g) + h \subseteq (f + g) \oplus h \subseteq (f \oplus g) \oplus h$.
Since $f \oplus (g \oplus h) = (f \oplus g) \oplus h$ it follows from the quasi-minimality of $(f \oplus g) \oplus h$ that
$f + (g + h) = (f + g) + h$.
For $f \in$ M(S, Z), let $-f = (-1)f$. Then by the definition (10) of addition in M(S, Z) it follows that $f + (-f) \subseteq f \oplus (-f)$
and since $0 \in f \oplus (-f)(s)$ for every $s \in S$, it follows that $f + (-f)(s) = \langle f \oplus (-f)\rangle (s) = \{0\}$.
For $\alpha, \beta \in$ K and $f \in$ M(S, Z), it follows that
$(\alpha + \beta)f \subseteq (\alpha f) \oplus (\beta f)$.
But $\alpha f + \beta f \subseteq (\alpha f) \oplus (\beta f)$ by (12) so that $\alpha f + \beta f = \langle(\alpha f) \oplus (\beta f)\rangle = (\alpha + \beta)f$.
It is notable that the algebraic operations on M(S, Z) are natural in at least two ways. At first, in terms of the quasilinear space Q(S, Z), the linear space M(S, Z) may be viewed as a quotient space with respect to the quasilinear subspace
$Q_0(S, Z) = \{f \in Q(S, Z) : 0 \in f (s), s \in S\}$
of Q(S, Z). Indeed, if $f , g \in Q(S, Z)$ then $f \oplus (-1 \odot g) \in Q0(S, Z)$ if and only if $\langle f \rangle = \langle g\rangle$. Therefore each $f \in$ M(S, Z) may be viewed as an equivalence class of quasi-minimal usco maps given by
$f + Q0(S, Z) = \{g \in Q(S, Z) : f - g \in Q0(S, Z)\} = \{g \in Q(S, Z) : \langle g\rangle = f \}$.
Then at second place, the algebraic operations of the M(S, Z) extend the usual point-wise operations on the set C(S, Z) of continuous functions from S into Z, in the way that the natural inclusion of C(S, Z) in M(S, Z) defines an injective linear transformation.

Topologies of Uniform Convergence on Compact Sets for Set-Valued Maps:

Consider Z *be* the metrizable topological vector space having translation invariant metric d_Z. The topology of uniform convergence on compact sets (τ_k) on M(S, Z) is defined in tandem with Hausdorff metric H on K(Z), is defined by the setting

$H(K, L) = \max\{\max\{d_Z (y, L) : y \in K\}, \max\{dZ (z, K) : z \in L\}\}$	(11)
for all $K, L \in$ K(Z). Here $d(z, K) = \min\{d_Z (z, y) : y \in K\}$ for all $K \in$ K(Z). For $f \in$ M(S, Z), $\epsilon > 0$ and $K \in$ K(S), let	
$W(f , K, \epsilon) = \{g \in M(S, Z) : H(f (s), g(s)) < \epsilon, s \in K\}$.	(12)

It's collection i.e. $\{W(f , K, \epsilon) : f \in M(S, Z), K \in K(S), \epsilon > 0\}$ is the basis for the "τ_k" on S. Following the previous operation we denote by $M_k(S, Z)$ the set of musco maps from S into Z *being* equipped with this topology. Based on the above theorem $M_k(S, R)$ is a locally convex linear topological space whenever S is locally compact. Vice versa, if S be a first countable, regular Baire space, then S is locally compact if and only if addition in $M_k(S, R)$ is continuous,
While $M_k(S, Z)$ is, in general, not a topological vector space, the collection $B_0 = \{W(0, K, \epsilon) : K \in K(S), \epsilon > 0\}$ is a basis at $0 \in$ M(S, Z) in terms with a vector space topology τ_s on M(S, Z). We call this topology τ_s the topology on *strong* uniform convergence on compact subsets of S, and denote M(S, Z) *being* equipped with this topology by $M_s(S, Z)$. As shown, here we invigilate some of the properties of $M_s(S, Z)$.

Properties of The Topological Vector Space M_s (S, Z):

Verifying our statement that B_0 is a basis at 0 for a vector space topology on M(S, Z).
Theorem 2 : *There is a Hausdorff vector space topology* τ_s *on* M(S, Z) *so that* B_0 *is a basis for the* τ_s-*neighbourhood filter at* $0 \in$ M(S, Z).
Proof. We prove that V_0 the filter generated by the collection B_0 satisfies the conditions .. To this end, fix $K \in$ K(S), $\epsilon > 0$ and $\alpha \in$ K \ {0}.

Clearly $0 \in W(0, K, \epsilon)$. Select $f , g \in W(0, K, \epsilon/2)$. If $s \in K$, $y \in f (s)$ and $z \in g(s)$ then, owing to the translation invariance of d_Z, it follows that $d_Z (y+z, 0) \leq d_Z (y, 0)+d_Z (z, 0) < \epsilon$. Therefore $f +g(s) \subseteq f \oplus g(s) \subseteq$

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$B(0, \epsilon)$ such that $H(\{0\}, f + g(s)) < \epsilon$ for all $s \in K$. Hence $f + g \in W(0, K, \epsilon)$ so that $W(0, K, \epsilon/2) + W(0, K, \epsilon 2) \subseteq W(0, K, \epsilon)$.
Accordingly there exists $\epsilon_\alpha > 0$ such that $B(0, \epsilon_\alpha) \subseteq \alpha B(0, \epsilon)$. If $f \in W(0, K, \epsilon_\alpha)$, then $f(s) \subseteq B(0, \epsilon\alpha) \subseteq \alpha B(0, \epsilon)$ so that $(1/\alpha) f(s) \subseteq B(0, \epsilon)$ for every $s \in K$. Hence $H(\{0\}, (1/\alpha) f(s)) < \epsilon$ for each $s \in K$, thus $(1/\alpha) f \in W(0, K, \epsilon)$. Hence
$f \in \alpha W(0, K, \epsilon)$ so that $W(0, K, \epsilon_\alpha) \subseteq \alpha W(0, K, \epsilon)$. If $h \in M(S, Z)$ then, since K is compact, $h(K)$ is compact. As such, there exists a constant $C > 0$ so that $\lambda h(s) \subseteq \lambda h(K) \subseteq B(0, \epsilon)$ for all $\lambda \in \mathrm{K}$ so that $|\lambda| < C$ and every $s \in K$. Thus $H(\{0\}, \lambda h(s)) < \epsilon$ for each $s \in K$ so that $\lambda h \in W(0, K, \epsilon)$ for each such . λ. That is, $W(0, K, \epsilon)$ is absorbing. It follows from previous operations that there exists $\epsilon' > 0$ so that $B(0, \epsilon)$ contains the balanced hull of $B(0, \epsilon')$. That is, $\{\alpha y : y \in B(0, \epsilon'), |\alpha| \leq 1\} \subseteq B(0, \epsilon)$. If $f \in W(0, K, \epsilon')$ then $f(s) \subseteq B(0, \epsilon')$ for every $s \in K$. Therefore $\alpha f(s)$ is contained in the balanced hull of $B(0, \epsilon')$, and hence in $B(0, \epsilon)$, for every $s \in S$ and $\alpha \in \mathrm{K}$ so that $|\alpha| \leq 1$. As a result $\alpha f \in W(0, K, \epsilon)$ whenever $f \in$
$W(0, K, \epsilon')$ and $|\alpha| \leq 1$ so that $W(0, K, \epsilon)$ contains the balanced hull of $W(0, K, \epsilon')$. Hence every element of V_0 contains an element of V_0 that is balanced. Hence all the conditions in the operation are satisfied so that V_0 is the neighbourhood filter at $0 \in M(S, Z)$ for a vector space topology τ_s on $M(S, Z$

Since

$$\bigcap_{\epsilon>0, K\in \mathrm{K}(E)} w(0, K, \epsilon) = \{0\}$$

it follows that τ_s is Hausdorff.

It is note worthy that the topology τ_k on $D(S, Z)$ depends on a particular metric d_Z on Z. It is also notable that the topology τ_k on $D(S, Z)$ depends on the particular metric d_Z on Z. That is, if d_1 and d_2 are metrics on Z *which is* compatible with the topology of Z, then the topology of uniform convergence on compact subsets of S generated by d_1 may differ from the one derived by the means of d_2. Holá showed that if S is a regular, first countable and non discrete, then two compatible metrics on Z generate the same topology of uniform convergence on compact subsets of S on $D(S, Z)$ if and only if the metrics are uniformly equivalent.

Theorem 3 : *If Z is locally convex, then* $M_s(S, Z)$ *is a locally convex space.*
Proof. Consider $K \in \mathrm{K}(S)$, $\epsilon > 0$, $f, g \in W(0, K, \epsilon)$ and $\alpha \in [0, 1]$. Then $f(s), g(s) \subseteq B(0, \epsilon)$ for all $s \in K$. If $y \in \alpha f + (1-\alpha) g(s)$ for some $s \in K$, then there exists $z \in f(s)$ and $w \in g(s)$ so that $y = \alpha z + (1-\alpha) w$. Hence $d_Z(y, 0) \leq d_Z((1-\alpha)w, 0) + d_Z(\alpha z, 0)$. But we may assume that the metric d_Z on Z satisfies $d_Z(\alpha v, 0) \leq |\alpha| d_Z(v, 0)$ for all $v \in Z$ and $\alpha \in \mathrm{K}$ with $|\alpha| \leq 1$. Hence $d_Z(y, 0) < \epsilon$. Since this holds for all $s \in K$ and $y \in \alpha f + (1 - \alpha) g(s)$ it follows that $\alpha f + (1 - \alpha) g \in W(0, K, \epsilon)$. Thus $M_s(S, Z)$ is a locally convex linear topological space.

Completeness and Metrizability Of M_s (S, Z):

We here, consider issues related to completeness and metrizability. We show that $M_s(S, Z)$ is complete, with respect to the natural uniformity induced by its vector space topology, whenever S is a locally compact Hausdorff space and Z is complete.
Furthermore, $M_s(S, Z)$ is metrizable if and only if S is hemicompact. Recall that S is hemicompact if there exists a countable subset K_0 of $\mathrm{K}(S)$ so that every $K \in \mathrm{K}(S)$ is contained in a member of K_0. Combining these results we see that if S is a locally compact Hausdorff space and Z is complete, then $M_s(S, Z)$ is completely metrizable if and only if S is hemicompact.
Theorem 4 : *If S is a locally compact Hausdorff space and Z is complete, then* $M_s(S, Z)$ *is a complete topological vector space.*
Proof. Consider $(f\gamma)_{\gamma \in \Gamma}$ be a Cauchy net in $M_s(S, Z)$. We state that, for every $s \in S$, $(f\gamma(s))_{\gamma \in \Gamma}$ is a Cauchy net in $\mathrm{K}(Z)$ *in terms* to the Hausdorff metric. Fix $\epsilon > 0$ and $s_0 \in S$. Since $(f\gamma)_{\gamma \in \Gamma}$ is a Cauchy net, there exists $\gamma_{s_0}^{\epsilon} \in \Gamma$ so that
$f_{\gamma 0} - f_{\gamma 1} \in W(0, \{s_0\}, \epsilon)$ whenever $\gamma_0, \gamma_1 \geq \gamma_{x_0}^{\epsilon}$. Fix $z_0 \in f_{\gamma 0}(s_0)$. Let

$$f'_{\gamma 0}(s) = \left\{ z \in f_{\gamma 0}(x) |^{\exists} \begin{array}{c} w \in f_{\gamma 1}(s) : \\ z - w \in f_{\gamma 0} - f_{\gamma 1}(s) \end{array} \right\}$$

for every $s \in S$. Since $f'_{\gamma 0}(x) \neq \emptyset$ for all $s \in S$ and $f_{\gamma 0}$ is musco, it follows that there exists a net $(s_\lambda)_{\lambda \in \Lambda}$ that converges to s_0 in S, and a net $(z_\lambda)_{\lambda \in \Lambda}$ converging to z_0 in Z so that $z_\lambda \in f'_{\gamma 0}(x_\lambda)$ for every $\lambda \in \Lambda$. Accordingly for $f'_{\gamma 0}$ there is, for each $\lambda \in \Lambda$, some $w_\lambda \in f_{\gamma 1}(s_\lambda)$ so that $z_\lambda - w_\lambda \in f_{\gamma 0} - f_{\gamma 1}(s_\lambda)$. There is a subnet of $(w_\lambda)_{\lambda \in \Lambda}$ that converges to some $w_0 \in f_{\gamma 1}(s_0)$. $z_0 - w_0 \in f_{\gamma 0} - f_{\gamma 1}(s_0)$. Therefore $d_Z(z_0, w_0) = d_Z(z_0 - w_0, 0) < \epsilon$. In the same manner it follows that for each $w_0 \in f_{\gamma 1}(s_0)$ there is $z_0 \in f_{\gamma 0}(x_0)$ so that $d_Z(z_0, w_0) < \epsilon$.Therefore $H(f_{\gamma 0}(s_0), f_{\gamma 1}(s_0)) < \epsilon$ whenever $\gamma_0, \gamma_1 \geq N^{\epsilon}_{s0}$, which verifies our statement.

Since Z is complete, it follows that we may assume that the metric on Y is complete. Therefore $\mathrm{K}(Z)$ is complete with respect to the Hausdorff metric. Hence $(f_\gamma(s))_{\gamma \in \Gamma}$ converges to some $K_s \in \mathrm{K}(Z)$, with respect to the Hausdorff metric, for every $s \in S$. It is clear from the preceding argument that the convergence of $(f_\gamma(s))_{\gamma \in \Gamma}$ to K_s is uniform on compact subsets of S. We state that the map $g : S \ni s \to K_s \in \mathrm{K}(Z)$ is quasi-minimal usco. Fix $s_0 \in S$ and a compact neighbourhood V_0 of s_0. Consider U be an open set containing $g(s_0)$. Since $g(s_0) = K_{s0}$ is compact, it follows from the Lebesgue Number Lemma that there exists $\epsilon > 0$ so that $g(s_0) \subseteq U_\epsilon(g(s_0)) \subseteq U$ where $U_\epsilon(K) = \bigcup\{B(y,\epsilon) : y \in K\}$ for any $K \in \mathrm{K}(Z)$. Since $(f_\gamma(s))_{\gamma \in \Gamma}$ converges uniformly on V_0 to $g(s)$ *in terms* with the Hausdorff metric, there is $\gamma_\epsilon \in \Gamma$ so that $f_\gamma(s) \subseteq U_{\epsilon/2}(g(s))$ and $g(s) \subseteq U_{\epsilon/2}(f_\gamma(s))$ whenever $s \in V_0$ and $\gamma \geq \gamma_\epsilon$. Fix $\gamma_0 \geq \gamma_\epsilon$. Since $f_{\gamma 0}$ is usco, there is $V \in \mathrm{V}_{s0}$ so that $f_{\gamma 0}(s) \subseteq U_{\epsilon/2}(g(s_0))$ wherever $s \in V$. Without loss of generality, we may assume that $V \subseteq V_0$ such that $U_{\epsilon/2}(f_{\gamma 0}(s)) \subseteq U_\epsilon(g(s_0))$ for every $s \in V$. Then $g(s) \subseteq U_\epsilon(g(s_0)) \subseteq U$ for all $s \in V$ such that g is usco at s_0. Since $s_0 \in S$ is arbitrary, it follows that g is usco on S. To see that g is quasi-minimal, consider some $\epsilon > 0$, $s_0 \in S$ and a compact neighbourhood V_0 of s_0.
Since $(f_\gamma(s))_{\gamma \in \Gamma}$ converges to $g(s)$ uniformly on V_0 there is $\gamma \in \Gamma$ such that $f_\gamma(s) \subset U\epsilon 3(g(s))$ and $g(s) \subset U_{\epsilon/3}(f_\gamma(s))$ for all $s \in V0$. It follows that there is $s_{V0} \in V_0$ so that diam $f_\gamma(s_{V0}) < \epsilon/3$. Since $g(s_{V0}) \subset U_{\epsilon/3}(f_\gamma(s_{V0}))$ it follows
that diam$(g(s_{V0})) < \epsilon$. Therefore the set $D_\epsilon = \{s \in S : \text{diam}(g(s)) < \epsilon\}$ is dense in S. Hence D_ϵ contains an open and dense set. Indeed, if $s \in D_{\epsilon/2}$ then there is $V \in \mathrm{V}_s$ such that $V \subseteq D_\epsilon$. Therefore g is quasi-minimal. It remains to show that $(f_\gamma)_{\gamma \in \Gamma}$ converges to $f = \langle g \rangle$ in $\mathrm{M}_s(S, Z)$. Fix $K \in \mathrm{K}(S)$ and $\epsilon > 0$. Consider K' be a compact subset of S containing K in its interior. Recall that $(f_\gamma(s))_{\gamma \in \Gamma}$ converges uniformly to $g(s)$ on K', with respect to the Hausdorff metric.
Since $f(s) \subseteq g(s)$ for every $s \in S$ it follows that there is $\gamma^{\varepsilon}_{K'} \in \Gamma$ such that $f(s) \subseteq U_{\epsilon/2}(f_\gamma(s))$ for every $s \in K'$ and $\gamma \geq \gamma^{\varepsilon}_{K'}$.
Therefore $f \oplus (-f_\gamma)(s) \cap B(0, (\epsilon/2)) \neq \emptyset$ for every $s \in K'$ and $\gamma \geq \gamma^{\varepsilon}_{K'}$. There is the map $h\gamma : C \Rightarrow Z$ defined as

$$h_\gamma(x) = \begin{cases} f \oplus (-f_\gamma)(x) \cap \bar{B}\left(0, \frac{\epsilon}{2}\right) & if\ x \in Int(K') \\ f \oplus (-f_\gamma)(x) & if\ x \notin (Int\ (K' \end{cases}$$

is usco. Since $h_\gamma \subseteq f \oplus (-f_\gamma)$ it follows that $h_\gamma \supseteq f - f_\gamma$. Therefore $f - f_\gamma(s) \subseteq B(0, \epsilon)$ for all $s \in K$ and $\gamma \geq \gamma^{\varepsilon}_{K'}$. Since this holds for all $K \in \mathrm{K}(S)$ and $\epsilon > 0$ it follows that $(f_\gamma)_{\gamma \in \Gamma}$ converges to f in $\mathrm{M}_s(S, Z)$.

Theorem 5 : $\mathrm{M}_s(S, Z)$ *is metrizable if and only if S is hemicompact.*
Proof. Assume that S is hemicompact. Then there exists a countable set $\mathrm{K}_0 = \{K_n : n \in \mathrm{N}\} \subseteq \mathrm{K}(S)$ so that every $K \in \mathrm{K}(S)$ is contained in a member of K_0. It is easy to see that $\{W(0, K_n, (1/m)) : m, n \in \mathrm{N}\}$ is a basis for τ_s at $0 \in \mathrm{M}_k(S, Z)$.
Therefore $\mathrm{M}_s(S, Z)$ is metrizable. Assume that $\mathrm{M}_s(S, Z)$ is metrizable. Since $\mathrm{C}(S, Z)$, with the usual point-wise operations, is a linear subspace of $\mathrm{M}(S, Z)$,
it follows that $\mathrm{C}(S, Z)$, as a topological subspace of $\mathrm{M}_s(S, Z)$, is a Hausdorff topological vector space. In fact, it is easy to see that the subspace topology on $\mathrm{C}(S, Z)$ is the compact open topology. It therefore follows that $\mathrm{C}(S, Z)$ is metrizable in the compact open topology so that S is hemicompact.

Corollary (a). *If S is a locally compact Hausdorff space and Z is complete, then* M*s(S, Z) is a completely metrizable if and only if S is hemicompact.*

Comparision Between $\mathrm{M}_s(S, Z)$ And $\mathrm{M}_k(S, Z)$:

Previously, we compared the topologies τ_s and τ_k. And showed that τ_s is finer than τ_k, with equality holding when *S* is locally compact. This result is then used, in combination with the results obtained in the preceding operations for $M_s(S, Z)$, to obtain corresponding results for $M_k(S, Z)$. We also explain how our results generalize the known results.

Theorem 6 : *The following statements are true.*
(i) *τ_s is finer than τ_k.*
(ii) *If S is locally compact, then $\tau_s = \tau_k$.*
Proof. (i) Consider $f \in M(S, Z)$, $K \in K(S)$, $\epsilon > 0$ and $g \in W(0, K, \epsilon)$. We state

that $f + g \in W(f, K, \epsilon)$. To see that this is so, consider an arbitrary point $s_0 \in K$ and $y_0 \in f + g(s_0)$. There is $z_0 \in f(s_0)$ and $w_0 \in g(s_0)$ such that $y_0 = z_0 + w_0$.
Then, since $g(s_0) \subset B(0, \epsilon)$, it follows that $d_Z(y_0, z_0) = d_Z(z_0 + w_0, z_0) = d_Z(w_0, 0) < \epsilon$. Now take $z_0 \in f(s_0)$. Let

$$f'(s) = \{ z \in f(s) \,|^{\exists}\, w \in g(s): z + w \in f + g(s)\}, \;\; s \in X$$

Since $f'(s) \neq \emptyset$ for every $s \in S$ and f is musco, there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ converging to x_0 in *S*, and a net $(z_\lambda)_{\lambda \in \Lambda}$ converging to z_0 in *Z* so that $z_\lambda \in f'(x_\lambda)$ for each $\lambda \in \Lambda$. By the definition of f' there exists, for each $\lambda \in \Lambda$, a point $w_\lambda \in g(x\lambda)$ so that $y_\lambda = z_\lambda + w_\lambda \in f + g(x_\lambda)$. Since *g* is usco, There exist $w_0 \in g(x_0)$ and a subnet of $(w_\lambda)_{\lambda \in \Lambda}$ that converges to w_0. There exists that $y_0 = z_0 + w_0 \in f + g(x_0)$. Since $g(x_0) \subset B(0, \epsilon)$, the translation invariance of the metric on *Z* implies that $d_Z(y_0, z_0) = d_Z(w_0, 0) < \epsilon$. Therefore $H(f(x_0), f + g(x_0)) < \epsilon$.
Since this is true for every $x_0 \in K$ our state has been proven so that $W(0, K, \epsilon) + f \subseteq W(f, K, \epsilon)$. Hence τ_s is finer than τ_k.
(ii) Now considering that *S* is locally compact. Consider $f \in M(S, Z)$, $K \in K(S)$ and $\epsilon > 0$. Let K' be a compact subset of *S* containing *K* in its interior. We state that $W(f, K', 2\epsilon) \subseteq W(0, K, \epsilon) + f$. It is sufficient to show that $h - f \in W(0, K, \epsilon)$
wherever $h \in W(f, K', (\epsilon/2))$. Since *S* is locally compact, and therefore a Baire space, it follows that the set $D = \{s \in S : \text{diam}\,(f(s)), \text{diam}\,(h(s)) = 0\}$ is dense in *S*. The above definition of addition in $M(S, Z)$ implies that $h - f(s) = h(s) - f(s)$ for every $s \in D$. Therefore $d_Z(h(s) - f(s), 0) < (\epsilon/2)$ for every $s \in D \cap K'$. It therefore that $h \oplus (-f)(S) \cap \bar{B}_{\varepsilon/2} \neq \emptyset$ for every $s \in \text{Int}(K')$. Hence the map $g : S \Rightarrow Z$ defined by

$$g(s) = \begin{cases} h \oplus (-f)(s) \cap \bar{B}_{\frac{\varepsilon}{2}}(0) & \text{if } s \in \text{Int}(K)' \\ h \oplus (-f)(s) & \text{if } s \notin \text{Int}(K)' \end{cases}$$

$$g(x) = \begin{cases} h \oplus (-f)(x) \cap \bar{B}_{\varepsilon/2}(0) & if\ x \in Int(K') \\ h \oplus (-f)(x) & if\ x \notin Int(K') \end{cases}$$

is usco. Since $h \oplus (-f)$ is quasi-minimal usco and $g, h\text{-}f \subseteq h \oplus (-f)$, it is that $h\text{-}f \subseteq g$. Hence $h - f(s) \subset B(0, \epsilon)$ such that $H(h - f(s), \{0\}) < \epsilon$ for every $s \in K$. Thus $h - f \in W(0, K, \epsilon)$.

Corollary (b). If S is locally compact, then the following statements are true.
(i) $M_k(S, Z)$ is a Hausdorff topological vector space.
(ii) The topology on $M_k(S, Z)$ is independent of the choice of compatible metric on Z.
(iii) If Z is locally convex, then $M_k(S, Z)$ is locally convex.
(iv) $M_k(S, Z)$ is metrizable if and only if S is hemicompact.
(v) If S is Hausdorff and Z is complete, then $M_k(S, Z)$ is complete.
(vi) If S is Hausdorff and Z is complete, then $M_k(S, Z)$ is completely metrizable if and only if S is hemicompact.

Holá explained that $M_k(S, Y)$ is completely metrizable wherever *Y* a complete metric space and *S* a locally compact hemicompact space. Corollary (b) (v) gives a partial converse to this result. Now considering that the

S is a Baire space, such that M$(S,$ R$) = D^*(S)$. This being the case, Corollary (b) (i) and (iii) generalize the above operations, while Corollary (b) (iv)–(vi) also partially generalizes it.

After satisfying the Generalization of the Concepts in terms os the Topology of Linear Spaces we proceed towards doing so for Quasilinear Spaces, as below:

Topological Quasilinear Spaces: Definitions and Properties

Allow us to start this by giving some concepts and basic results. For some topological space S, the N_s stands denoting the family of all neighborhoods of an $s \in S$. Consider S be a topological vector space TVS, for short, $s \in S$ and $G \subset S$. Then $G \in N_s$ if and only if $G - s \in N_0$ and $s - G \in N_0$. This is what called the localization principle of TVSs.

A set S is called a *quasilinear space* QLS, for short, if a partial ordering relation "≤", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such way that the following conditions hold for any elements $s, y, z, u \in S$, and any real scalars α, β:

$s \leq s,$

$s \leq z \quad$ if $s \leq y, y \leq z,$

$s=y \quad$ if $s \leq y, y \leq s,$

$s + y= y + s,$

$s + (y + z) = (s + y) + z$

there is an element $0 \in S$ such that $s \; 0 \; s,$

$\alpha \cdot (\beta \cdot s) = (\alpha \cdot \beta) \cdot s,$

$\alpha \cdot (s + y) = \alpha \cdot s + \alpha \cdot y$

$1 \cdot s = s,$

$0 \cdot s = 0,$

$(\alpha + \beta) \cdot s \leq \alpha \cdot s + \beta \cdot s,$

$s + z \leq y + v \quad$ if $s \leq y, z \leq v,$

$\alpha \cdot s \leq \alpha \cdot y \quad$ if $s \leq y.$

A linear space is a QLS with the partial ordering relation "$s \leq y$ if and only if $s = y$". Perhaps the most popular example of nonlinear QLSs is the set of all closed intervals of real numbers with the inclusion relation "⊆" , algebraic sum operation

$A+ B = \{a \; b : a \in A, b \in B\}$ (13)

and the real-scalar multiplication

$\lambda A = \{\lambda a : a \in A\}$ (14)

Denoting this set by KCR. While another one is KR, the set of all compact subsets of real numbers. In general, KE and KCE stand for the space of all nonempty closed bounded and nonempty convex and closed bounded subsets of any normed linear space E, respectively. Both are QLSs with the inclusion relation and with a slight modification of addition as follows:

$A + B = \overline{\{a \; b : \; a \in A, b \in B\}}$ (15)

and with real-scalar multiplication above.

Hence, $K_C(E) = \{A \in KE : A$ convex$\}$.

Lemma 1 *In a* QLS *S the element* 0 *is minimal, that is,* $s = 0$ *if* $s \leq 0$.

Definition . An element $s \in S$ is called an inverse of an $s \in S$ if $s + s' = 0$. If an inverse element exists, then it is unique. An element s having an inverse is called regular; otherwise, it is called singular.

We prove later that the minimality is not only a property of 0 but also is shared by the other regular elements.

Lemma 2. *Suppose that each element s in the QLS S has an inverse element* $s' \in S$*. Then the partial ordering in S is determined by equality, the distributivity conditions hold, and, consequently, S is a linear space.*

Corollary (c) : *In a real linear space, equality is the only way to define a partial ordering such that operation (13) hold.*

It will be assumed in what follows that $-s = -1(s)$. An element s in a QLS is regular if and only if $s - s = 0$ if and only if $x' = -s$.

Definition . Suppose that S is a QLS and $Y \subseteq S$. Y is called a subspace of S wherever Y is a quasilinear space with the same partial ordering and the same operations on S.

Theorem 7 : *Y is a subspace of a QLS S if and only if for every $s, y \in Y$ and $\alpha, \beta \in$ R, $\alpha s + \beta y \in Y$.*

Proof of this theorem is quite similar to its classical linear algebraic counterpart.

Consider S be a QLS and Y be a subspace of S. Suppose that each element s in Y has an inverse element $x' \in Y$; then the partial ordering on Y is determined by the equality. In this case the distributivity conditions hold on Y, and Y is a linear subspace of S.

Definition. *Consider S* be a QLS. An element $s \in S$ is said to be symmetric provided that $-s = s$, and X_b denotes the set of all such elements. Further, X_r and X_s stand for the sets of all regular and singular elements in S, respectively.

Theorem 8 : *X_r, X_d, and $X_s \cup \{0\}$ are subspaces of S.*

Proof. X_r is a subspace since the element $\lambda x' + y'$ is the inverse of $\lambda s + y$.

$X_s \cup \{0\}$ is a subspace of S. Let $s, y \in X_s \cup \{0\}$ and $\lambda \in$ R. The assertion is clear for $s= y= 0$. Let $s \neq 0$ and suppose that $s+ \lambda y \notin Xs \cup \{0\}$, that is, $(s+ \lambda y)+ u = 0$ for some $u \in S$. Then $s+(\lambda y+ u)= 0$ and so $x' = \lambda y + u$. This implies that $s \in X_r$. Analogously we obtain $y \in X_r$ if $y \neq 0$. This contradiction shows that $s + \lambda y \in X_s \cup \{0\}$.

The proof for X_d is similar.

X_r, X_d, and $X_s \cup \{0\}$ are called *regular*, *symmetric*, and *singular subspaces* of S, respectively.

Example. Consider $S = K_C$R and $Z= \{0\} \cup \{a, b : a, b \in$ R and $a \neq b\}$. Z is the singular subspace of S. However, the set $\{\{a\} : a \in$ R$\}$ of all singletons constitutes X_r and is a linear subspace of S. Factually, for any normed linear space E, each singleton $\{a\}$, $a \in E$ is identified with a, and hence E is considered as the regular subspace of both KCE and KE.

Lemma 3 The operations of algebraic operations of addition and scalar multiplication are continuous with respect to the Hausdorff metric. The norm is continuous function with respect to the Hausdorff metric.

Lemma 4 *(a) Suppose that $s_n \to s_0$ and $y_n \to y_0$, and that $s_n \leq y_n$ for any positive integer n. Then $s_0 \leq y_0$.*

(b) Suppose that $s_n \to s_0$ and $z_n \to s_0$. If $s_n \leq y_n \leq z_n$ for any n, then $y_n \to s_0$.

(c) Suppose that $s_n + y_n \to s_0$ and $y_n \to 0$; then $s_n \to s_0$.

Example Consider S be a real complete normed linear space a real Banach space. Then S is a complete normed quasilinear space with partial ordering given by equality. Conversely, if S is complete normed quasilinear space and any $s \in S$ has an inverse element $x' \in S$, then S is a real Banach space, and the partial ordering on S is the equality. In this case $h_x(x, y) = \quad \| x - y \|_x$, it is notable that $h_x(x, y) \neq \| x - y \|_x$ for nonlinear QLS, in general.

For example, if E is a Banach space, then a norm on $K(E)$ is defined by $\| A \|_{K(E)} = sup \| a \|_E$. Then KE and K_CE are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:

$$,h(A, B) = \inf\{r \geq 0 : A \subseteq B + S_r(0), B \subseteq A + S_r(0\}) \qquad (15)$$

Defining the Topology of Quasilinear Spaces:

Definition. A topological quasilinear space TQLS, for short S is a topological space and a quasilinear space such that the algebraic operation of addition and scalar multiplication are continuous, and, following conditions are satisfied for any $s, y \in S$:

for any $U \in$ N0, $s \leq y$ and $y \in U$ implies $s \in U$, (16)

for any $U \in N_s$, $y \in U \Longleftrightarrow$ there IS some $V \in$ N0 satisfying $s+V \subseteq U$,

such that $s \leq y+a$ for some $a \in V$ or $y \leq s+b$ for some $b \in V$. (17)

Any topology τ, which makes S, τ be a topological quasilinear space, will be called a *quasilinear topology*. The

ABOVE conditions provide necessary harmony of the topology with the ordering structure on S.

Example *Consider S* be a TVS. Then, for any $s, y \in S$ and for any $U \in N_s$, $y \in U$ if and only if there exists a neighborhood V of 0 satisfying $s+V \subseteq U$ such that $s= y + a$ for some $a \in V$ or $y= s+ b$ for some $b \in V$. In fact, this is true by the localization principle of TVSs since $U - s$ and $s - U$ are neighborhood of 0. Hence we obtain desired V by taking $V= U - s$ or $V= s - U$. This provides the previous conditions. Hence, S is a TQLS. We later show that some TQLS may *not* satisfy the localization principle.

Remark. In condition 3.2, for some $U \in N_s$, we may find a $V \in N_0$ satisfying $s+V \subseteq U$ such that both $s \leq y+a$ and $y \leq s + b$ for some $a, b \in V$. This comfortable situation depends on the selection of U. However, we may not find such a suitable $V \in N_0$ for some $U \in N_x$ even in TVS.

Esample. Consider real numbers with usual metric. Take $s =3$, $y= 5$, and $U = [2,7] \in N_s$. Then any $V \in N_0$ satisfying $3+V \subseteq U$ must be a subset of $[-1, 4]$. Further $3 =5+a$ and $5= 3+b$ gives $a = -2$, $b= 2$, and hence V can only include b.

Remark . In a semimetrizable TQLS the condition can be reformulated by balls as follows:
for any $\varepsilon > 0$, $s \leq y$ and $y \in S\varepsilon 0$ implies $s \in S_\varepsilon(0)$ (18)
equivalently,

$s \leq y$ implies $d(s, 0) \leq d(y, 0)$,
for any $\varepsilon > 0$, $y \in S_\varepsilon$ s⇐⇒ there exists some $S\varepsilon 0$,
with s $S\varepsilon 0 \subseteq S\varepsilon s$ such that $s \leq y$ a for some $a \in S_\varepsilon 0$,
or $y \leq s+b$ for some $b \in S_\varepsilon 0$. (19)

A TQLS with a semimetrizable quasilinear topology will be called a *(semi)metric QLS*.

Theorem 9 : *Let S be a* TQLS. *Then X_r and X_d are closed in S.*
Proof. $\{s_i\}$ is a net in S_r converging to an $s \in S$. By the continuity of algebraic operations $-s_i \to -s$ and $s_i - s_i \to s - s$. This means $s - s= 0$ since $s_i - s_i = 0$ for each i, whence $s \in S_r$.
The proof is easier for X_d.
The result of this theorem may not be true for $X_s \cup \{0\}$. Consider $S = K_C(R)$ and define $x_n = [1, 1\ \frac{1}{n}] \in X_s \cup \{0\}$ for each $n \in N$.

Then $s_n \to \{1\} \notin X_s \cup \{0\}$.

Definition Consider S be a quasilinear space. A paranorm on S is a function p : S → R satisfying the following conditions. For every s, y ∈ S,
(i) p(0) = 0

(ii) p(s) ⩾ 0

(iii) p(-s) = p(s)

(iv) p(s+y) ⩽ p(s) + p(y)

(v) if $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{s_n\} \subset S$ with $p(s_n) \to p(s)$, then $p(t_n s_n) \to p(ts)$ and (continuity of scalar multiplication)
(vi) if $s \leq y$, then $p(s) \leq p(y)$.
The pair (S, p) with the function p satisfying the conditions (i) to (vi) is called a paranormed QLS. There exist that if any $s \in S$ has an inverse element $x' \in S$, then the concept of paranormed quasilinear space coincides with the concept of a real paranormed linear space. This paranorm is called *total* if, in addition, we have $p(s) = 0 \Longleftrightarrow s$ 0,

if for any $\varepsilon > 0$ there exists an element $s_\varepsilon \in S$ such that, $s \le y + s_\varepsilon$ and $p(s_\varepsilon) \le \varepsilon$, then $s \le y$.

(20)

The equality

$$d(x,y) = \inf\{r \ge 0 : x \le y + a_1^r, y \le x + a_2^r, p(a_i^r) \le r\} \qquad (21)$$

defines a semimetric on a paranormed quasilinear space S. d is metric wherever p is total.

Now, consider p be total and $d(s, y) = 0$. Then for any $\varepsilon > 0$ there exist elements$x_\varepsilon^1, x_\varepsilon^2 \in S$ such that $s \le y + x_\varepsilon^1$, and $y \le s + x_\varepsilon^2$, for $p(x_\varepsilon^i) \le \varepsilon$, i 1, 2. Hence the totality conditions imply that $s \le y$ and $y \le s$, that is, s y.

Further, we have the inequality $d(s+, y) \le p(s - y)$.
Note that these definitions are inspired from the definitions about normed
quasilinear spaces. The proofs of some facts given here are quite similar to that of Aseev's corresponding results.
If the first condition in the definition of norm in a QLS is relaxed into the condition

$\|s\| \ge 0$ if $s \ne 0$ (22)

and if the norm is removed, we obtain the definition of a seminorm.
A quasilinear space having seminorm is called a seminormed QLS. Similarly in linear spaces it can proved that a seminorm on a QLS is a paranorm. Thus we have the following implication chain among the kinds of QLS :
normed-seminormed QLS ⇒ total paranormed-paranormed QLS ⇒ metric-semimetric QLS ⇒ Hausdorff TQLS.

Definition. *Consider (S, d)* be a semimetric QLS and s be an element of S. Then, the nonnegative number

$\rho(s) = d(s - s, 0)$ (23)

is called diameter of s. For every regular element s, $\rho(s) = 0$ since $s - s =$ 0. Hence this definition is redundant in linear spaces. In addition it should not be confused with the classical notion of the diameter of a subset in a semimetric space for which it is defined by $\delta(U) = \sup_{s,y\in U} d(s, y)$ for any $U \subset S$.
For example, in K_C(R),[-1, 3] ∈ KCR and $\rho([-1, 3]) = h([-1, 3] - [-1, 3], 0)$

$= h([-4,4],0) = \|[-4,4]\|$

$= \sup |a| = 4$

$a \in [-4,4]$. (24)

However, for the singleton subset U = {[-1, 3]} of K_C(R), $\delta(U) = 0$.
result is half of the localization principle of TVS.

Theorem 10: Consider *S be a TQLS, s ∈ S, and U is a set containing 0. If s + U ∈* N_s,, *then U ∈* N_0.
Proof. The proof is just an application of the fact that the translation operator $fs : S \to S$, $f_s(v) = v + s$, is continuous by the continuity of the algebraic sum operation. Though the converse of this theorem is true in almost all the TVSs, it may not be true in some TQLSs.

Example Consider K_C(R) again and let it be closed unit ball $S_1(0)$. Now, for s = [2, 3] ∈ K_C(R), we show that $s + S_1(0)$ is not the neighborhood of s. A careful observation shows that $s + S_1(0)$ do not contain elements intervals for which the diameter is smaller than 1. However, every s-centered ball $S_r(s)$ with radius r contains a singleton if $r \ge \rho(s)/2$ = ½ and contains an interval such as [2+(r/2), 3 –(r/2)] if r < 1/2 since h([2, 3] ,[2+($r/2$), 3 – ($r/2$)]=(r.2)<r (25)
That shows that, $S_r(s)$ contains elements with diameter smaller than 1. However, neither a singleton nor such an element belongs to $s + S_r(0)$. This implies, $S_r(s) \not\subseteq s + S_r(0)$ for every $r > 0$. Eventually, the set $s + S_1(0)$ cannot contain an s-centered ball. Thus, the localization principle may not be satisfied about a singular element in K_C(R). The example points that translation by a singular element destroys the property of being a neighborhood in a TQLS.

Theorem 11: Consider *S be a TQLS and* $s \in X_r$. *Then* $U \in N_0 \Leftrightarrow s + U \in N_s$.
Proof. Consider again the operator f_s in the Theorem 10. If this be the case the inverse f_s^{-1} exists and is the continuous operator f_{-s}. Hence f_s is a homeomorphism hence preserves the neighborhoods.

CONCLUSIONS

This paper has significantly advanced the understanding of topological properties within the framework of quasilinear spaces and their crucial applications to set-valued maps. Our work successfully generalizes Aseev's foundational concept of normed quasilinear spaces to the more encompassing domain of topological quasilinear spaces, thereby broadening the theoretical landscape for such structures.

A central contribution is the rigorous demonstration that the space of minimal semi-continuous compact-valued maps, M(S,Z), can be endowed with a linear space structure, even when S is an arbitrary topological space and Z a metrizable topological vector space. This finding is particularly impactful as it extends previous results that were confined to more restrictive conditions on S, such as being a Baire space, necessitating the development of novel techniques based on the metric characterization of quasi-minimal upper semi-continuous compact-valued maps.

Furthermore, the paper provides a detailed topological analysis of M(S,Z) when equipped with the topology of strong uniform convergence on compact subsets of S, denoted as $M_s(S,Z)$. We establish several fundamental properties of this space, including its completeness under the conditions that S is a locally compact Hausdorff space and Z is complete. A key result is the characterization of its metrizability, proving that $M_s(S,Z)$ is metrizable if and only if S is hemicompact. This comprehensive characterization provides essential tools for further research into the analytical properties of such function spaces.

The comparative study of the strong uniform topology (τ_s) and the topology of uniform convergence on compact sets (τ_k) reveals that τ_s is consistently finer than τ_k, with these topologies coinciding when S is locally compact. This elucidates the precise relationship between different modes of convergence in this generalized setting. Moreover, our investigation into the principle of localization highlights that its universal validity, typical in topological vector spaces, may not extend to all topologically quasilinear spaces, especially concerning singular elements. This nuance is critical for avoiding pitfalls in future theoretical developments.

Collectively, these findings offer fresh and profound insights into the intricate topology of quasilinear spaces, significantly enriching the fields of multivalued mappings and set-valued analysis. The established linear and topological structures provide a robust foundation for advancing functional analysis in non-linear contexts, paving the way for new applications in areas dealing with uncertainty and set-valued phenomena.

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