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# Using a General Hurwitz-Lerch Zeta for BI-Univalent Analytic Functions to Estimate a Second Hankel Determinant

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## ABSTRACT

In this paper, we introduce and investigate a new class of bi- univalent functions defined in the open unit disk U involving a general integral operator associated with the general Hurwitz- Lerch Zeta function denoted by $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ . The main result of the investigation is to estimate the upper bounds for the initial Taylor–Maclaurin coefficients of functions $|a_2|$ and $|a_3|$ for this class. Following, we find the second Hankel determinant. Several new results are shown after specializing the parameters employed in our main results.

**Keywords-** Hankel determinant, Bi-univalent functions, coefficient bounds, Hurwitz -Lerch zeta function.

## INTRODUCTION

Let A denote the class of all analytic functions in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ , and normalized by the conditions $f(0) = f'(0) - 1 = 0$, and given by the power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \qquad (1.1)$$

Let S denote the subclass of A consisting of univalent functions. The well-known Koebe one-quarter theorem (see [1]) every univalent function $f \in S$ contains a disk of radius $\left(\frac{1}{4}\right)$, the inverse of $f \in U$ is a univalent analytic function on the disk $U_p \coloneqq \left\{z: z \in \mathbb{C} \text{ and } |z| < p; p \geq \frac{1}{4}\right\}$.
Therefore, for each function $f(z) = w \in S$, there is an inverse function $f^{-1}(w)$ of $f(z)$ defined by

$$f^{-1}(f(z)) = z \ (z \in U) \text{ and } f(f^{-1}(w)) = w \ (w \in U_p),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \ . \qquad (1.2)$$

If $f$ and $f^{-1}$ are univalent function in U, then we say the function f is bi-univalent function in U.

The class of bi-univalent function in U given by (1.1) denoted by Σ. The following are some important examples of bi-univalent functions in U

$$\Sigma \frac{z}{1-z}, \quad \log\frac{1}{1-z} \quad \text{and} \quad \log\sqrt{\frac{1+z}{1-z}}.$$

Lewin [8] investigated in 1967 and showed a bound of the coefficient on the class Σ of bi-univalent functions and estimated $|a_2| < 1.5$. Following, Brannan and Clunie [9] showed the result of Lewin and established that $|a_2| < \sqrt{2}$. Afterwards, Netanyahu [10] showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$. Then many authors approximate the Taylor-

Maclaurin coefficient $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ and defined new subclasses of bi-univalent analytic functions unit disk see (e.g. [11]- [13]). However, the problem to estimate the coefficients of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ still an open. In current paper, employing the techniques of Srivastava et al [13] which have brought back interest in the study of analytic and bi-univalent functions, we introduce a new class and estimates of the coefficients $|a_2|$ and $|a_3|$, although we estimate a second Hankel determinant for a class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ .

In 1976 Noonan and Thomas [2] defined the Hankel determinant of a function f for $q \geq 1$ and $k \geq 1$

$$H_q(k)=\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.$$

The determinant $H_q(k)$ has been extensively studied with $H_2(2)$ referring to the second Hankel determinant which is defined by $|a_2a_4 - a_3^2|$. It has also been investigated by several authors (e.g. [3]-[7]).

**Definition.1.1** [14] A general Hurwitz–Lerch Zeta function $\Phi\,(z,s,b)$ defined by

$$\Phi\,(z,s,b)=\sum_{k=0}^{\infty}\frac{z^k}{(k+b)^s},$$

where $(s \in \mathbb{C}, b \in \mathbb{C}-\mathbb{Z}_0^-)$ when $(|z|<1)$ , and $(\Re(s)>1)$ when $(|z|=1)$.

Nagat and Darus [17], introduced the generalized integral operator associated with the general Hurwitz- Lerch Zeta function, denoted by $\mathfrak{J}_{s,b}^{\alpha}f(z)$ for $f \in A$ as follows:

For $(s \in \mathbb{C}, b \in \mathbb{C}-\mathbb{Z}_0^-)$ the generalized integral operator $\mathfrak{J}_{s,b}^{\alpha}f(z)\colon A \to A$ is defined by

$$\mathfrak{J}_{s,b}^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}\Phi(z,s,b),\quad (\alpha \neq 2,3,4,\dots)$$

$$= z+\sum_{k=2}^{\infty}\varphi_k^{\alpha,b,s}a_kz^k\,,(z\in U)\,, \tag{1.3}$$

where

$$\varphi_k^{\alpha,b,s}=\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(\frac{b}{k+1-b}\right)^s.$$

Many other works on analytic and univalent functions related to this operator can be see (e.g. [15],[17], [18]).

By using a generalized integral operator, a new class of bi-univalent functions are considered as the following.

**Definition.1.1:** For $s \in \mathbb{C}, b \in \mathbb{C}-\mathbb{Z}_0^-$ and $\alpha \neq 2,3,4,\dots$, a function $f \in \Sigma$ and of the form (1.1) is said to be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ if the following conditions are satisfied:

$$\Re\left[\frac{(1-\lambda)\left(\mathfrak{J}_{s,b}^{\alpha}f(z)\right)}{z}+\lambda\left(\mathfrak{J}_{s,b}^{\alpha}f(z)\right)^{`}\right]>\beta,(0\leq\beta<1\,,\lambda\geq 1\,,z\in U),$$

and

$$\Re\left[\frac{(1-\lambda)\left(\mathfrak{J}_{s,b}^{\alpha}g(w)\right)}{w}+\lambda\left(\mathfrak{J}_{s,b}^{\alpha}g(w)\right)^{`}\right]>\beta,(0\leq\beta<1\,,\lambda\geq 1\,,z\in U),$$

where $g=f^{-1}$ .

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It is of interest to note that by taking $\alpha = 0$ and $s = 0$ in Definition 1.1, we state the following class $\mathcal{B}_{\sum}(\beta, \lambda)$ due to Frasin et al. [19] in the next remark.

**Remark 1:** A function $f \in \sum$ and of the form (1.1) is said to be in the class $\mathcal{B}_{\sum}(\beta, \lambda)$ if the following conditions are satisfied:

$$\Re\left[\frac{(1-\lambda)(f(z))}{z} + \lambda\big(f(z)\big)^{`}\right] > \beta, \quad (0 \le \beta < 1, \lambda \ge 1, z \in U),$$

and

$$\Re\left[\frac{(1-\lambda)(g(w)}{w} + \lambda\big(g(w)\big)^{`}\right] > \beta, \quad (0 \le \beta < 1, \lambda \ge 1, z \in U),$$

where $g = f^{-1}$ .

It is of interest to note that by taking $\alpha = 0$ ,$s = 0$ and $\lambda = 1$ in Definition1.1, we state the following class $\mathcal{H}_{\sum}(\beta, \lambda)$ due to Srivastava et al. [20] in the next remark.

**Remark 2:** A function $f \in \sum$ and of the form (1.1) is said to be in the class $\mathcal{H}_{\sum}(\beta, \lambda)$ if the following conditions are satisfied:

$$\Re[f'(z)] > \beta, \quad (0 \le \beta < 1, z \in U),$$

and

$$\Re[g'(w)] > \beta, \quad (0 \le \beta < 1, z \in U),$$

where $g = f^{-1}$.

## SULTS

In order to derive our main results, we have to recall here the following lemmas

**Lemma 2.1.**[1] Let $\mathcal{P}$ be the class of all analytic functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

satisfying $\Re\big(p(z)\big) > 0 (z \in U)$and $p(0) = 1$.Then $|p_n| \le 2, (n = 1,2,3, \dots)$.

**Lemma 2.2.** [21] if the function $p \in P$ is given by the series

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2 z),$$

for some $x, z$ with $|x| \le 1$ and $|z| \le 1$.

**Lemma 2.3.** [22] The power series for $p$ given in (2.1) converges in U to a function in $P$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & & \cdots & c_{-n+2} \end{vmatrix}, \quad n = 1,2,3, \dots$$

And $c_{-k} = \overline{c_{k,}}$ and all non-negative. They are strictly positive except for

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$p(z)=\sum_{k=1}^{m} p_k p_0\left(e^{it_k z}\right), p_k>0,\ \ t_k\in\Re$ and $t_k\neq t_j$ for $k\neq j$ in this case $D_n>0$ for $n<m-1$ and $D_n=0$ for $n\geq m$.

## MAIN RESULTS

**Theorem 3.1** Let f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$, Then

$$|a_2|\leq\sqrt{\frac{2(1-\beta)}{(1+2\lambda)\varphi_3^{\alpha,b,s}}}.$$

$$|a_3|\leq\frac{4(1-\beta)^2}{(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2}+\frac{2(1-\beta)}{(1+2\lambda)\varphi_3^{\alpha,b,s}}.$$

Proof: Since $f\in\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ there exists two analytic functions ρ and $q: U\rightarrow U$ with $\rho(0)=q(0)=0$ satisfying the following conditions.

$$(1-\lambda)\frac{\mathfrak{J}_{s,b}^{\alpha}f(z)}{z}+\lambda\left(\mathfrak{J}_{s,b}^{\alpha}f(z)\right)'=\beta+(1-\beta)\rho(z),\qquad(3.1)$$

and

$$(1-\lambda)\frac{\left(\mathfrak{J}_{s,b}^{\alpha}g(w)\right)}{w}+\lambda\left(\mathfrak{J}_{s,b}^{\alpha}g(w)\right)`=\beta+(1-\beta)q(w),\qquad(3.2)$$

Define the functions p(z) and q(w) by

$$\rho(z)=1+c_1z+c_2z^2+c_3z^3+\cdots\qquad(3.3)$$

And

$$q(w)=1+d_1w+d_2w^2+d_3w^3+\cdots\qquad(3.4)$$

Applying (3.3) and (3.4) in (3.1) and (3.2), respectively

$$1+\frac{(1+\lambda)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2z+\frac{(1+2\lambda)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s a_3z^2+\frac{(1+3\lambda)\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)}\left(\frac{b}{b+3}\right)^s a_4z^3+\cdots,$$

$$=1+(1-\beta)c_1z+(1-\beta)c_2z^2+(1-\beta)c_3z^3+\cdots,\qquad(3.5)$$

and

$$1-\frac{(1+\lambda)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2w+\frac{(1+2\lambda)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s(2a_2^2-a_3)w^2+\frac{(1+3\lambda)\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)}\left(\frac{b}{b+3}\right)^s(5a_2^3-5a_2a_3+a_4)w^4+\cdots,$$

$$=1+(1-\beta)d_1w+(1-\beta)d_2w^2+(1-\beta)d_3w^3+\cdots.\qquad(3.6)$$

Now, by comparing the coefficients in (3.5), we see that:

$$\frac{(1+\lambda)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2=(1-\beta)c_1,\qquad(3.7)$$

$$\frac{(1+2\lambda)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s a_3=(1-\beta)c_2,\qquad(3.8)$$

$$\frac{(1+3\lambda)\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)}\left(\frac{b}{b+3}\right)^s a_4=(1-\beta)c_3, \tag{3.9}$$

And by comparing the coefficients in (3.6), we see that:

$$\frac{-(1+\lambda)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2=(1-\beta)d_1, \tag{3.10}$$

$$\frac{(1+2\lambda)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s (2a_2^2-a_3)=(1-\beta)d_2, \tag{3.11}$$

$$\frac{-(1+3\lambda)\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)}\left(\frac{b}{b+3}\right)^s (5a_2^3-5a_2a_3+a_4)=(1-\beta)d_3. \tag{3.12}$$

From (3.7) and (3.10), gives

$$a_2=\frac{(1-\beta)}{(1+\lambda)\varphi_2^{\alpha,b,s}}c_1=-\frac{(1-\beta)}{(1+\lambda)\varphi_2^{\alpha,b,s}}d_1=-d_1 \tag{3.13}$$

$$c_1=-d_1,$$

and

$$2(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2a_2^2=(1-\beta)^2(c_1^2+d_1^2). \tag{3.14}$$

Also, from (3.8) and (3.11), we get:

$$2(1+2\lambda)\varphi_3^{\alpha,b,s}a_2^2=(1-\beta)(c_2+d_2).$$

Thus, we have

$$|a_2|^2\le\frac{(1-\beta)(|c_2|+|d_2|)}{2(1+2\lambda)\varphi_3^{\alpha,b,s}}\le\frac{2(1-\beta)}{(1+2\lambda)\varphi_3^{\alpha,b,s}},$$

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{(1+2\lambda)\varphi_3^{\alpha,b,s}}}.$$

Subtracting (3.8), (3.11), we get

$$2(1+2\lambda)a_3\varphi_3^{\alpha,b,s}-2(1+2\lambda)a_2^2\varphi_3^{\alpha,b,s}=(1-\beta)(c_2-d_2),$$

$$a_3=a_2^2+\frac{(1-\beta)(c_2-d_2)}{2(1+2\lambda)\varphi_3^{\alpha,b,s}},$$

Upon substituting the value of $a_2^2$ from (3.14), we obtain

$$a_3=\frac{(1-\beta)^2(c_1^2+d_1^2)}{2(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2}+\frac{(1-\beta)(c_2-d_2)}{2(1+2\lambda)\varphi_3^{\alpha,b,s}},$$

Applying lemma 2.1

$$|a_3|\le\frac{4(1-\beta)^2}{(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2}+\frac{2(1-\beta)}{(1+2\lambda)\varphi_3^{\alpha,b,s}}.$$

As applications of Theorem3.1 about upper bounds for coefficients $a_2$and $a_3$for the analytic and bi-univalent functions in the new class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$, we obtain and improve the known results by [19] in the following corollaries by setting the particular values of the parameters $\alpha$ , $s, \lambda$ *and* $\varphi_k^{\alpha,b,s}$.

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**Corollary 1.** ([19]) Let f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ ,$0 \le \beta < 1$ ,$\lambda \ge 1$ by taking $\alpha = 0$ , $s = 0$, $\varphi_2^{\alpha,b,s} = 1$, and $\varphi_3^{\alpha,b,s} = 1$. Then the upper bounds for two initial coefficients are

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(1+2\lambda)}},$$

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(1+2\lambda)}.$$

**Corollary 2.** [20] Let f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda) 0 \le \beta < 1$ ,$\lambda \ge 1$ by taking $\alpha = 0$ , $\lambda = 1$, $s = 0$, $\varphi_2^{\alpha,b,s} = 1$, and $\varphi_3^{\alpha,b,s} = 1$. Then the upper bounds for two initial coefficients are

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3}},$$

$$|a_3| \le \frac{(1-\beta)(5-3\beta)}{3}.$$

**Theorem 3.2 Let** f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda), 0 \le \beta < 1$ ,$\lambda \ge 1$ , Then the upper bound for the Second Hankel is

$$|a_2a_4 - a_3^2| \le \begin{cases} 4(1-\beta)^2\left[\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 + 4(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}}{(1+\lambda)^4(1+3\lambda)(\varphi_2^{\alpha,b,s})^4\varphi_4^{\alpha,b,s}}\right]; \beta \in \left[0, 1 - \sqrt{\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{8(1+3\lambda)\varphi_4^{\alpha,b,s}}}\right], \\ \frac{9(1+\lambda)^2(1-\beta)^2\left(\varphi_2^{\alpha,b,s}\right)^2}{2(1+3\lambda)\varphi_4^{\alpha,b,s}\left[(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 - 2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}\right]}; \beta \in \left(1 - \sqrt{\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{8(1+3\lambda)\varphi_4^{\alpha,b,s}}}, 1\right). \end{cases}$$

Proof: since $f \in \mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ and from (3.13) in Theorem3.1, we get:

$$c_1 = -d_1.$$

From (3.8) and (3.11) and using (3.13), we get:

$$(1+2\lambda)\varphi_3^{\alpha,b,s}(2a_2^2 - a_3) - (1+2\lambda)\varphi_3^{\alpha,b,s}a_3 = (1-\beta)(c_2 - d_2).$$

Then

$$(1+2\lambda)\varphi_3^{\alpha,b,s}a_3 - 2(1+2\lambda)\varphi_3^{\alpha,b,s}\frac{(1-\beta)^2}{(1+\lambda)^2(\varphi_2^{\alpha,b,s})^2}c_1^2 + (1+2\lambda)\varphi_3^{\alpha,b,s}a_3 = (1-\beta)(c_2 - d_2).$$

$$2(1+2\lambda)\varphi_3^{\alpha,b,s}a_3 = (1-\beta)(c_2 - d_2) + 2(1+2\lambda)\varphi_3^{\alpha,b,s}\frac{(1-\beta)^2}{(1+\lambda)^2(\varphi_2^{\alpha,b,s})^2}c_1^2$$

$$a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2}c_1^2 + \frac{(1-\beta)}{2(1+2\lambda)\varphi_3^{\alpha,b,s}}(c_2 - d_2). \qquad (3.15)$$

Also, from (3.9) and (3.12), and using (3.13), we get:

$$-(1+3\lambda)\varphi_4^{\alpha,b,s}(5a_2^3 - 5a_2a_3 + a_4) - (1+3\lambda)\varphi_4^{\alpha,b,s}a_4 = (1-\beta)(c_2 - d_2)$$

$$-5(1+3\lambda)\varphi_4^{\alpha,b,s}a_2^3+5(1+3\lambda)\varphi_4^{\alpha,b,s}a_2a_3-(1+3\lambda)\varphi_4^{\alpha,b,s}a_4-(1+3\lambda)\varphi_4^{\alpha,b,s}a_4$$

$$=(1-\beta)(c_3-d_3)$$

$$2(1+3\lambda)\varphi_4^{\alpha,b,s}a_4-\frac{5(1+3\lambda)(1-\beta)^2\varphi_4^{\alpha,b,s}}{2(1+2\lambda)(1+\lambda)\varphi_2^{\alpha,b,s}\varphi_3^{\alpha,b,s}}c_1(c_2-d_2)=(1-\beta)(c_3-d_3)$$

$$a_4=\frac{1}{2(1+3\lambda)\varphi_4^{\alpha,b,s}}\left[\frac{5(1+3\lambda)(1-\beta)^2\varphi_4^{\alpha,b,s}}{2(1+2\lambda)(1+\lambda)\varphi_2^{\alpha,b,s}\varphi_3^{\alpha,b,s}}c_1(c_2-d_2)+(1-\beta)(c_3-d_3)\right]. \qquad (3.16)$$

From (3.13), (3.15) and (3.16), we stabilize that

$$|a_2a_4-a_3^2|=\left|-\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4}c_1^4+\frac{(1-\beta)^3}{4(1+2\lambda)(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2\varphi_3^{\alpha,b,s}}c_1^2(c_2-d_2)+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c_1(c_3-d_3)-\frac{(1-\beta)^2}{4(1+2\lambda)^2\left(\varphi_3^{\alpha,b,s}\right)^2}(c_2-d_2)^2\right|.$$

According to lemma2.2, we have

$$2c_2=c_1^2+x(4-c_1^2),$$

and

$$2d_2=d_1^2+x(4-d_1^2),$$

then, $c_2=d_2$,

and further

$$4c_3=c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2z),$$

$$4d_3=d_1^3+2(4-d_1^2)d_1x-d_1(4-d_1^2)x^2+2(4-d_1^2)(1-|x|^2z)$$

$$c_3-d_3=\frac{1}{2}c_1^3+c_1(4-c_1^2)x-\frac{1}{2}c_1(4-c_1^2)x^2,$$

$$|a_2a_4-a_3^2|=\left|-\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4}c_1^4+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c_1(c_3-d_3)\right|,$$

$$|a_2a_4-a_3^2|=\left|-\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4}c_1^4+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c_1\left(\frac{1}{2}c_1^3+c_1(4-c_1^2)x-\frac{1}{2}c_1(4-c_1^2)x^2\right)\right|$$

$$|a_2a_4-a_3^2|=\left|-\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4}c_1^4+\frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c_1^4+\frac{(1-\beta)^2c_1^2(4-c_1^2)}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}x-\frac{(1-\beta)^2c_1^2(4-c_1^2)}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}x^2\right|.$$

Now letting $c=c_1$, where $c\in[0,2]$ with $\mu=|x|\leq 1$, we obtain:

$$|a_2a_4-a_3^2|\leq\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4}c^4+\frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c^4+\frac{(1-\beta)^2c^2(4-c^2)}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}\mu+\frac{(1-\beta)^2c^2(4-c^2)}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}\mu^2=F(\mu).$$

Differentiating $F(\mu)$ ,we get:

$$F'(\mu) = \frac{(1-\beta)^2c^2(4-c^2)}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}} + \frac{(1-\beta)^2c^2(4-c^2)}{2(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}\mu.$$

By using elementary calculus, one can show that $F'(\mu) > 0$ for $\mu > 0$, hence F is an increasing function and thus, the upper bound for $F(\mu)$ corresponds to $\mu = 1$ ,in which case

$$F(\mu) = F(1) = \left[\frac{(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4} + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}\right]c^4 + \frac{3(1-\beta)^2}{4(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}c^2(4-c^2) = G(c).$$

Assuming $G(c)$ has a maximum value in an interior of $c \in [0,2]$,by elementary calculations, we find

$$G'(c) = \left[\frac{4(1-\beta)^4}{(1+\lambda)^4\left(\varphi_2^{\alpha,b,s}\right)^4} - \frac{2(1-\beta)^2}{(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}\right]c^3 + \frac{6(1-\beta)^2c}{(1+\lambda)(1+3\lambda)\varphi_2^{\alpha,b,s}\varphi_4^{\alpha,b,s}}.$$

Then $G'(c) = 0$ implies that real's critical point $c_{01} = 0$ or $c_{02} = \sqrt{\frac{3(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 - 2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}}}$ .

Now we will find the value of $\beta$,

$$\beta = 1 - \sqrt{\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{8(1+3\lambda)\varphi_4^{\alpha,b,s}}}.$$

We came to the following conclusions after some calculations:

**Case (1):** when $\beta \in \left[0, 1 - \sqrt{\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{8(1+3\lambda)\varphi_4^{\alpha,b,s}}}\right]$ ,we observe that $c_{02} \geq 0$ , is $c_{02}$ is out of the interval $(0,2)$, therefore the maximum value of $G(c)$ occurs at $c_{01} = 0$ or $c = c_{02}$ which contradiction our assumption of having the maximum value at the interior point of $c \in [0,2]$. Since $G(c)$ is an increasing function in the interval $[0,2]$ , maximum point of G must be on the boundary of $c \in [0,2]$ that is $c = 2$.

Thus, we have:

$$\max_{0\le c\le 2} G(c) = G(2) = 4(1-\beta)^2\left[\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 + 4(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}}{(1+\lambda)^4(1+3\lambda)(\varphi_2^{\alpha,b,s})^4\varphi_4^{\alpha,b,s}}\right].$$

**Case (2):** when $\beta \in \left(1 - \sqrt{\frac{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{8(1+3\lambda)\varphi_4^{\alpha,b,s}}}, 1\right)$ ,we observe that $c_{02} \leq 2$ that is $c_{02}$ is an interior of the interval $[0,2]$ so the maximum value $G(c)$ occurs at $c = c_{02}$.Thus we have

$$\max_{0\le c\le 2} G(c) = G(c_{02}) = G\left(\sqrt{\frac{3(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3}{(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 - 2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}}}\right)$$

$$= \frac{9(1+\lambda)^2(1-\beta)^2\left(\varphi_2^{\alpha,b,s}\right)^2\left[(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s} - (1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3\right]}{2(1+3\lambda)\varphi_4^{\alpha,b,s}\left[(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 - 2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}\right]^2} + \frac{3(1-\beta)^2(1+\lambda)^2\left(\varphi_2^{\alpha,b,s}\right)^2}{(1+3\lambda)\varphi_4^{\alpha,b,s}\left[(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3 - 2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}\right]}$$

$$=\frac{9(1+\lambda)^2(1-\beta)^2\left(\varphi_2^{\alpha,b,s}\right)^2}{2(1+3\lambda)\varphi_4^{\alpha,b,s}\left[(1+\lambda)^3\left(\varphi_2^{\alpha,b,s}\right)^3-2(1-\beta)^2(1+3\lambda)\varphi_4^{\alpha,b,s}\right]}.$$

This completes the proof of Theorem 3.2.

In particular, Theorem 3.2 gives the following corollaries

By preferring $\alpha=0$ , $s=0$, $\varphi_2^{\alpha,b,s}=1$ , $\varphi_3^{\alpha,b,s}=1$ and $\varphi_4^{\alpha,b,s}=1$, in Theorem3.2

**Corollary 1.** ([23]) Let f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ ,$0\leq\beta<1$ ,$\lambda\geq1$ by taking $\alpha=0$ , $s=0$, $\varphi_2^{\alpha,b,s}=1$ , $\varphi_3^{\alpha,b,s}=1$ and $\varphi_4^{\alpha,b,s}=1$, Then the upper bound for the Second Hankel is

$$|a_2a_4-a_3^2|\leq\begin{cases}4(1-\beta)^2\left[\frac{(1+\lambda)^3+4(1-\beta)^2(1+3\lambda)}{(1+\lambda)^4(1+3\lambda)}\right]; & \beta\in\left[0\ \ 1-\sqrt{\frac{(1+\lambda)^3}{8(1+3\lambda)}}\right],\\ \frac{9(1+\lambda)^2(1-\beta)^2}{2(1+3\lambda)[(1+\lambda)^3-2\,(1-\beta)^2(1+3\lambda)]}; & \beta\in\left(1-\sqrt{\frac{(1+\lambda)^3}{8(1+3\lambda)}},1\right).\end{cases}$$

**Corollary 2.** ([24]) Let f be an analytic and bi-univalent function given by (1.1) be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\beta,\lambda)$ ,$0\leq\beta<1$ ,$\lambda\geq1$ by taking $\alpha=0$ ,$\lambda=1$, $s=0$, $\varphi_2^{\alpha,b,s}=1$ , $\varphi_3^{\alpha,b,s}=1$ and $\varphi_4^{\alpha,b,s}=1$, Then the upper bound for the Second Hankel is

$$|a_2a_4-a_3^2|\leq\begin{cases}(1-\beta)^2\left[(1-\beta)^2+\frac{1}{2}\right]; & \beta\in\left[0\ ,\frac{1}{2}\right],\\ \frac{9(1-\beta)^2}{16[1-(1-\beta)^2]}, & \beta\in\left(\frac{1}{2},1\right).\end{cases}$$

# CONCLUSION

In this study, A new class of bi-univalent functions in the open unit disk has been introduced and defined via a general integral operator. Derived estimates for the initial coefficients of functions and further obtained an upper bound for the second Hankel determinant for this class. Several existing and new results may be identified as special cases of our main result. By varying the parameters involved, these results contribute to the further development of the theory of bi-univalent functions and open new avenues for future studies of other special functions operators. Many research papers have been utilized to investigate various problems related to this area, can be see (e.g. [25-26]).

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