Periodic Patterns and Block Structures in Squared Pell Sequence Modulo 10^e

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ABSTRACT

In this paper, we investigate the periodic properties of the squared Pell sequence $\{SP_n\}$, which is defined by the recurrence relation $SP_n = P_n^2$; for all $n \geq 2$; with $P_0 = 0, P_1 = 1$, where P_n denotes n^{th} Pell number. For any modulus $m > 1$, we introduce a novel concept of 'blocks' within this sequence by examining the distribution of residues over a single period of the squared Pell sequence. Our results reveal that the length of any given period of the squared Pell sequence comprises either 1 or 2 blocks.

Keywords: Fibonacci sequence, Pell sequence, Periodicity of Pell sequence.

INTRODUCTION

The Fibonacci sequence $\{F_n\}$ shows interesting periodic properties under modulo 10^e. Initially, the last digits of Fibonacci numbers seem random, but a clear pattern emerges: the sequence of last digits repeats every 60 numbers. Therefore, the last digits exhibit a periodicity with a cycle length of 60, expressed as $F_{60n+i} \equiv F_i (mod\ 10)$ for any i, where $n \geq 0$. Koshy [7] proved this using mathematical induction.

In 1972, Kramer and Hoggatt Jr. [2] established the periodicity of Fibonacci sequence as well as of Lucas sequence when considered modulo 10^n. Patel, Shah [5] considered the periodicity of generalized Lucas numbers and proved the result when the length of its period under modulo 2^e.

This brings in to mind an immediate question - For any given positive integer $m > 1$, does the sequence $\{F_n\}$ is periodic when considered modulo m? In 1960, Wall [6] examined the periodic nature of $\{F_n\}$ with respect to any positive integer $m > 1$ and showed that $\{F_n\}$ consistently exhibits periodicity.

Ömür Deveci, Erdal Karaduman [3] proved some elementary results for the periodicity of $\{P_n\}$. For further details about Pell numbers, one can refer Horadam [1] and Koshy [8].

This listing can be further extended as several articles are available in the literature concerning the periodicity of varied generalizations of the Fibonacci sequence. In the following section, we now consider the periodicity of a new sequence – the squared Pell sequence.

SQUARED PELL SEQUENCE

The *squared Pell sequence* is the sequence which consists of the squares of all the Pell numbers in order.

Definition: The sequence $\{SP_n\}$ represents the squares of corresponding terms of the sequence $\{P_n\}$ in order. In other words, $SP_n = P_n^2$; for all $n \geq 1$, where P_n stands for n^{th} Pell number.

It is trivial to note that $\{SP_n\} = \{0, 1, 4, 25, 144, 841, 4900, 28561, 166464, \ldots\}$. We first derive some elementary results for this sequence which will be used further in this paper. The following result gives a recurrence relation which helps to reduce the terms of $\{SP_n\}$ into smaller terms.

Lemma 2.1:

$SP_{m+n} = SP_m SP_{n+1} + SP_{m-1} SP_n + 2\, P_m P_n P_{m-1} P_{n+1}$.

Lemma 2.2: $SP_{2n} = SP_n(SP_{n-1} + SP_{n+1}) + 2\, SP_n\, P_{n-1}\, P_{n+1}$.

Lemma 2.3: $SP_{2n+1} = SP_n^2 + SP_{n+1}^2 + 2\, SP_n SP_{n+1}$.

In the following section we study the periodicity of sequence $\{SP_n\}$ and obtain some interesting results related with its residues.

PERIODICITY OF SQUARED PELL SEQUENCE

In this section, we study in detail about the periodic nature of $\{SP_n\}$ when considered modulo $m > 1$. For the detailed insights, one can refer Marc [9].

Definition: By $SP(mod\, m)$, we mean the sequence of the least non-negative residues of the terms of the squares of terms of the sequence $\{P_n\}$ in order taken modulo m.

As an illustration, we consider $SP(mod\ 8)$ in the following table:

Table 4.1: $SP(mod\ 8)$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**
SP_n	0	1	4	25	144	841	4900
$SP(mod\ 8)$	0	1	4	1	0	1	4

From the above table, it can be noticed that the sequence $SP(mod\ 8)$ is periodic. Furthermore, it is not difficult to check that $SP_{4n+i} \equiv P_i(mod\ 8)$; where $n \geq 0$. This clearly indicates that the period of $SP(mod\ 8)$ is 4.

We now prove several results for the periodic nature of $SP(mod\, m)$ analogues to that of $P(mod\, m)$.

Lemma 3.1: The sequence $SP(mod\, m)$ is always periodic; for any integer $m > 1$ and its starting values $0, 1$.

We next introduce the notation for the length of period of $SP(mod\, m)$.

Definition: $k_{SP}(m)$ denotes the length of period of the squared Pell sequence modulo m.

The following are some immediate consequences from the lemmas 4.6.1, 4.6.2 and the definition of $k_{SP}(m)$.

Lemma 3.2: (a) $SP_{k_{SP}(m)-2} \equiv 4\ (mod\, m)$

(b) $SP_{k_{SP}(m)-1} \equiv 1\ (mod\, m)$

(c) $SP_{k_{SP}(m)} \equiv 0\ (mod\, m)$

(d) $SP_{k_{SP}(m)+1} \equiv 1\ (mod\, m)$

(e) $SP_{k_{SP}(m)+2} \equiv 4\ (mod\, m)$

(f) $SP_{k_{SP}(m)+3} \equiv 25\ (mod\, m)$

(g) $SP_{k_{SP}(m)+nr} \equiv P_n(mod\, m), \forall\ r \in \mathbb{Z}$.

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Fact 3.3: Since $SP(mod\ m)$ is periodic, we will often use the fact that 'if both $SP_n \equiv 0(mod\ m)$ and $SP_{n+1} \equiv 1\ (mod\ m)$ holds, then $k_{SP}(m) \mid n$.

Lemma 3.4: For any given integer m, there are infinitely many squared Pell numbers which are divisible by m.

Theorem 3.5: If $n \mid m$ then $k_{SP}(n) \mid k_{SP}(m)$.

Theorem 3.6: $k_{SP}(m) = lcm[k_{SP}(\,p_i{}^{e_i})]$, for various values of i, where $m = \prod p_i{}^{e_i}$ and p_i's are distinct primes.

Theorem 3.7: $k_{SP}(lcm[m,n]) = lcm[k_{SP}(m), k_{SP}(n)]$.

VALUE OF $k_{SP}(2^e)$

In this section, we obtain the value of $k_{SP}(p^e)$ when $p = 2$.

Theorem 4.1: $k_{SP}(2^e) = \begin{cases} 3 & ; e = 1 \\ 2^{e-1} & ; e \geq 2 \end{cases}$.

Proof: We notice that $SP(mod\ 2) = \{0, 1, 1, 0, 1, \ldots\}$. Therefore, $k_{SP}(2) = 3$. For $e \geq 2$, we prove the result by induction.

We note that $SP(mod\ 4) = \{0, 1, 0, 1, \ldots\}$ and $SP(mod\ 8) = \{0, 1, 4, 1, 0, 1, \ldots\}$. Therefore, $k_{SP}(4) = 3$ and $k_{SP}(8) = 6$. This proves the result for $e = 2, 3$. We assume that the result holds for some positive integer $e = r \geq 4$. Thus,

$$k_{SP}(2^r) = 2^{r-1}; r \geq 4. \qquad (1)$$

Then by the lemma 3.2 $(a), (b)$ and (c), we have

$$\left.\begin{array}{c} SP_{2^{r-1}} \equiv 0(mod\ 2^r); SP_{2^{r-1}+1} \equiv 1(mod\ 2^r) \\ SP_{2^{r-1}-1} \equiv 1\ (mod\ 2^r) \end{array}\right\} \qquad (2)$$

By lemma 2.2, we have $SP_{2n} = SP_n(SP_{n-1} + SP_{n+1}) + 2\,SP_n P_{n-1} P_{n+1}$. By taking $n = 2^{r-1}$ and using (2), we have

$$SP_{2\times 2^{r-1}} = SP_{2^{r-1}}(SP_{2^{r-1}-1} + SP_{2^{r-1}+1}) + 2\,SP_{2^{r-1}} P_{2^{r-1}-1} P_{2^{r-1}+1}$$

$$\equiv 0 \times (1+1) + 2 \times 0 \times 1 \times 1(mod\ 2^{r+1})$$

Thus,

$$SP_{2^r} \equiv 0(mod\ 2^{r+1}) \qquad (3)$$

Again, by lemma 2.3, we have $SP_{2n+1} = SP_n^2 + SP_{n+1}^2 + 2SP_n SP_{n+1}$. Considering $n = 2^{r-1}$, we get

$$SP_{2\times 2^{r-1}+1} = SP_{2^{r-1}}^2 + SP_{2^{r-1}+1}^2 + 2\,SP_{2^{r-1}} SP_{2^{r-1}+1}. \quad (4)$$

But by (1), we get $SP_{2^{r-1}} \equiv 0(mod\ 2^r)$ and $SP_{2^{r-1}+1} \equiv 1(mod\ 2^r)$. Thus, $SP_{2^{r-1}} = 0, 2^r, 2 \times 2^r, 3 \times 2^r, \ldots$ and $SP_{3\times 2^{r-2}+1} = 1, 1 + 2^r, 1 + 2 \times 2^r, \ldots$. By considering modulo 2^{r+1}, we have $SP_{2^{r-1}} = 0$ or 2^r; and $SP_{2^{r-1}} \equiv 1$ or $1 + 2^r$. Thus by (4), we have

$$SP_{2^r+1} \equiv (0 \text{ or } 2^r)^2 + \left(1 \text{ or}(1+2^r)\right)^2 + 2(0 \text{ or } 2^r)\left(1 \text{ or } (1+2^r)\right)(mod\ 2^{r+1})$$

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$\equiv (0 \text{ or } 2^{2r}) + (1 \text{ or } 1 + 2^{r+1} + 2^{2r}) + (2^{r+1} \text{ or } 2^{r+1} + 2^{2r+1}) (mod\ 2^{r+1})$

Thus,

$SP_{2^r+1} \equiv 1 (mod\ 2^{r+1})$ (5)

Then by the $(3), (5)$ and fact 3.3, we have

$k_{SP}(2^{r+1}) \mid 2^r$ (6)

Since $2^r \mid 2^{r+1}$ implies $k_{SP}(2^r) \mid k_{SP}(2^{r+1})$, we get

$2^{r-1} \mid k_{SP}(2^{r+1})$ (7)

Then by combining equation (6) and (7), we get

$k_{SP}(2^{r+1}) = 2^{r-1}$ or $k_{SP}(2^{r+1}) = 2 \times 2^{r-1} = 2^r$.

We shall show that the case $k_{SP}(2^{r+1}) = 2^{r-1}$ is not possible. In fact, we will show that $SP_{2^{r-1}+1} \not\equiv 1 (mod\ 2^{r+1})$. More precisely, we will prove that

$SP_{2^{r-1}+1} \equiv 1 + 2^r (mod\ 2^{r+1}); r \geq 4.$ (8)

Considering $r = 4$, we have (i) $SP_9 = 970225 \equiv 17 = 1 + 2^4 (mod\ 2^5)$ and (ii) $SP_{17} = 1292061882721 \equiv 33 = 1 + 2^5 (mod\ 2^6)$. Therefore, (8) is true for $r = 4$. Let it be true for some integer $r - 1$. Thus, $SP_{2^{r-2}+1} \equiv 2^r + 1\ (mod\ 2^r)$. Considering modulo 2^{r+1}, we get

$SP_{2^{r-2}+1} \equiv (2^{r-1} + 1)$ or $(2^{r-1} + 1 + 2^r)$ (9)

Then, $SP^2_{2^{r-2}+1} \equiv (2^{r-1} + 1)^2$ or $(2^{r-1} + 1 + 2^r)^2 (mod\ 2^{r+1})$.

Now since $r \geq 4$, we have $(2^{r-1} + 1)^2 = 2^{2r-2} + 2^r + 1 \equiv 2^r + 1 (mod\ 2^{r+1})$.

Also, $(2^{r-1} + 1 + 2^r)^2 = 2^{2r-2} + 2^{2r+1} + 2^{r+1} + 2^r + 1 \equiv 2^r + 1 (mod\ 2^{r+1})$.

This gives,

$SP^2_{2^{r-2}+1} \equiv 2^r + 1 (mod\ 2^{r+1})$. (10)

We also assume that

$SP_{2^{r-2}} \equiv 0 (mod\ 2^r)$ (11)

(This is because if it is not true then replacing r by $r + 1$, we can say that $SP_{2^{r-1}} \equiv 0 (mod\ 2^{r+1})$ is not true. Thus, $k_{SP}(2^{r+1}) \neq 2^{r-1}$ which we need to prove.) Taking modulo 2^{r+1}, we get $SP_{2^{r-2}} \equiv 0$ or 2^r. Thus

$SP^2_{2^{r-2}} \equiv 0 (mod\ 2^{r+1})$ (12)

Now, by lemma 2.3, we have $SP_{2n+1} = SP_n^2 + SP_{n+1}^2 + 2\, SP_n \times SP_{n+1}$. Considering $n = 2^{r-2}$, we get

$SP_{2^{r-1}+1} = SP^2_{2^{r-2}} + SP^2_{2^{r-2}+1} + 2\, SP_{2^{r-2}} \times SP_{2^{r-2}+1}$ (13)

By $(9), (10), (11)$ and (12), we thus have

$SP_{2^{r-1}+1} \equiv 0 + (2^r + 1) + \quad 2(0 \text{ or } 2^r)(2^{r-1} + 1 \text{ or } 2^{r-1} + 1 + 2^r)(mod\ 2^{r+1}) \equiv\ 2^r + 1 + (2^{2r} + 2^{r+1} \text{or} 2^{2r} + 2^{r+1} + 2^{2r+1})(mod\ 2^{r+1})$

Thus, $SP_{2^{r-1}+1} \equiv 2^r + 1(mod\ 2^{r+1})$ (14)

This now confirms that $SP_{2^{r-1}+1} \not\equiv 1(mod\ 2^{r+1})$; that means $k_{SP}(2^{r+1}) = 2^{r-1}$ is not possible. Hence $k_{SP}(2^{r+1}) = 2^r$. This proves the theorem by induction.

VALUE OF $k_{SP}(5^e)$

In this section, we obtain the value of $k_{SP}(p^e)$ for the case $p = 5$.

Theorem 5.1: $k_{SP}(5^e) = 6 \times 5^{e-1}; e \geq 1$.

Proof: To prove the required result, it is sufficient to prove that

$$\left.\begin{array}{r} SP_{6\times 5^{e-1}} \equiv SP_0 \equiv 0(mod\ 5^e) \\ SP_{6\times 5^{e-1}+1} \equiv SP_1 \equiv 1(mod\ 5^e). \end{array}\right\} \qquad (15)$$

We use induction to prove these results. For $e = 1$, we have

$SP_{6\times 5^{1-1}} \equiv SP_6 = 4900 \equiv 0(mod\ 5^e)$ and $SP_{6\times 5^{1-1}+1} \equiv SP_7 = 28561 \equiv 1(mod\ 5^e)$.

Thus, (15) is true for $e = 1$. We next assume that results hold for some positive integer $e = r \geq 2$. That is, let the following holds:

$k_{SP}(5^r) = 6 \times 5^{r-1}$ (16)

We prove that (15) holds for $e = r + 1$ also. Therefore, we need to prove that $SP_{6\times 5^r} \equiv SP_0 \equiv 0(mod\ 5^{r+1})$ and $SP_{6\times 5^r+1} \equiv SP_1 \equiv 1(mod\ 5^{r+1})$. Now, by lemma 1.2.3 (e), we have $SP_n = 5P_{2n+3} - SP_{n+3}$. By considering $n = 6 \times 5^r$, we get

$SP_{6\times 5^r} = 5 \times P_{2\times 6\times 5^r+3} - SP_{6\times 5^r+3}$.

Also, by Koshy [2,3], we have $P_{12\times 5^r+3} \equiv 5\ (mod\ 5^{r+1})$ and by lemma 3.2 (f), we have $SP_{6\times 5^r+3} \equiv 25\ (mod\ 5^{r+1})$. Therefore, $SP_{6\times 5^r} = 5 \times P_{12\times 5^r+3} - SP_{6\times 5^r+3} \equiv 5 \times 5 - 25(mod\ m)$. Thus,

$SP_{6\times 5^r} \equiv 0(mod\ 5^{r+1})$ (17)

By Koshy [3], we have $P_{2n+1} = P_n^2 + P_{n+1}^2$.

By considering $n = 6 \times 5^r$, we get $SP_{6\times 5^r+1} = P_{12\times 5^r+1} - SP_{6\times 5^r}$. Now, by lemma 4.1.1 (d), we have $P_{12\times 5^r+1} \equiv 1(mod\ 5^{r+1})$. Also, by lemma 2.3 (c), we have $SP_{6\times 5^r} \equiv 0(mod\ 5^{r+1})$. Therefore,

$SP_{6\times 5^r+1} = P_{12\times 5^r+1} - SP_{6\times 5^r} \equiv 1 - 0\ (mod\ 5^{r+1})$

Thus,

$SP_{6\times 5^r+1} \equiv 1(mod\ 5^{r+1})$ (18)

Using the (17), (18) and fact 3.3, we can now conclude that

$k_{SP}(5^{r+1}) \mid 6 \times 5^r$ (19)

Since $5^r \mid 5^{r+1}$ implies that $k_{SP}(5^r) \mid k_{SP}(5^{r+1})$. Also, by induction hypothesis, we have $k_{SP}(5^r) = 6 \times$

5^{r-1}. This gives

$$6 \times 5^{r-1} \mid k_{SP}(5^{r+1}) \qquad (20)$$

Thus using (19) and (20), we conclude that

$k_{SP}(5^{r+1}) = 6 \times 5^{r-1}$ or $k_{SP}(5^{r+1}) = 6 \times 5^{r}$.

We finally confirm that the case $k_{SP}(5^{r+1}) = 6 \times 5^{r-1}$ is not possible. In fact, we show that $SP_{6\times5^{r-1}} \not\equiv 0(mod\ 5^{r+1})$. Now by Koshy [3], we get

$$SP_n = \frac{1}{2}\{2SP_{n+1} - P_{2n+1} - P_{2n} - (-1)^n\}.$$

Considering $n = 6 \times 5^{r-1}$, we get

$$SP_{6\times5^{r-1}} = \frac{1}{2}\left\{2 \times SP_{6\times5^{r-1}+1} - P_{12\times5^{r-1}+1} - P_{12\times5^{r-1}} - (-1)^{12\times5^{r-1}}\right\}.$$

Since by Koshy [3], we have $P_{12\times5^{r-1}} \equiv 0\ (mod\ 5^r)$ and $P_{12\times5^{r-1}+1} \equiv 1\ (mod\ 5^r)$ and thus in modulo 5^{r+1}, we have $P_{12\times5^{r-1}} \equiv 0$ or 5^r and $P_{12\times5^{r-1}+1} \equiv 1$ or $1 + 5^r$. Also, by lemma 2.3 (c) and (d), we have $SP_{6\times5^{r-1}} \equiv 0(mod\ 5^r)$ and $SP_{6\times5^{r-1}+1} \equiv 0(mod\ 5^r)$.

Thus, in modulo 5^{r+1}, we have $SP_{6\times5^{r-1}} \equiv 0$ or 5^r and $SP_{12\times5^{r-1}+1} \equiv 1$ or $1 + 5^r$. We get

$$SP_{6\times5^{r-1}} = \frac{1}{2}\left\{\begin{matrix}2 \times SP_{6\times5^{r-1}+1} - P_{12\times5^{r-1}+1} - P_{12\times5^{r-1}} - \\ (-1)^{12\times5^{r-1}}\end{matrix}\right\}$$

$$\equiv \frac{1}{2}\{(1 \text{ or } 1 + 5^r) - (1 \text{ or } 1 + 5^r) - (0 \text{ or } 5^r) - 1\}$$

$$\equiv \frac{1}{2}\{5^r - 1\}(mod\ 5^{r+1})$$

Therefore, $SP_{6\times5^{r-1}} \not\equiv 0(mod\ 5^{r+1})$. This shows that $k_{SP}(5^{r+1}) = 6 \times 5^{r-1}$ is not possible. Hence, $k_{SP}(5^{r+1}) = 6 \times 5^{r}$. Thus, $k_{SP}(5^{e}) = 6 \times 5^{e-1}$ is true for every positive integer e, which proves the required result.

Finally, using theorem 4.1, 5.1 and 3.7, we easily conclude the following important result.

Theorem 5.2: $k_{SP}(10^e) = \begin{cases} 6 & ; e = 1 \\ 3 \times 10^{e-1} & ; e \geq 2 \end{cases}$.

The following result calculates the period of $\{P_n\}$ when considered modulo 10^e.

Theorem 5.3: $SP_{6t+n} \equiv SP_n(mod\ 10)$ and $SP_{3\times10^{e-2}t+n} \equiv SP_n(mod\ 10^e)$; where $e \geq 2, n > 0$ and t is any integer.

In the next section, we introduce the notion of blocks within the period of the squared Pell sequence.

BLOCKS WITHIN THE PERIOD OF SQUARED SEQUENCE

In this final section, we study the nature of the blocks within the residues of the squared Pell sequence when considered modulo m. We also discuss the distribution of residues within a single period of $SP(mod\ m)$. For the detailed insights, one can refer Patel, Shah [4].

Definition: $\alpha_{SP}(m)$ denotes the smallest positive value of index n of squared Pell numbers such that $SP_n \equiv 0(mod\ m)$ and $SP_{n-1} = SP_{n+1}$; when $n > 1$.

Thus, $SP_{\alpha_{SP}(m)} \equiv 0(mod\ m)$. We call $\alpha_{SP}(m)$ to be the *restricted period* of $SP(mod\ m)$. Thus $\alpha_{SP}(m)$ indicates the position of ending of first block which occurs in $SP(mod\ m)$. We call the finite sequence SP_0, $SP_1, \dots$, $SP_{\alpha_P(m)-1}$ to be the *first block* occurring in $SP(mod\ m)$.

Definition: When $\alpha_{SP}(m) = k_{SP}(m)$, we call $SP(mod\ m)$ to be *without restricted period*.

To illustrate these definitions, we consider the following examples.

(i) Since $SP(mod\ 3) = \{0, 1, 1, 1, 0, 1, 1, 1\ \dots\}$, then clearly $\alpha_{SP}(3) = k_{SP}(3) = 4$. In this case $SP(mod\ 3)$ will be without restricted period.
(ii) Since $SP(mod\ 5) = \{0, 1, 4, 0, 4, 1, 0, 1, \dots\}$, we have $k_{SP}(5) = 6$ and $\alpha_{SP}(5) = 3$. Thus, $SP_3 \equiv 0(mod\ 5)$. Here $0, 1, 4$ is the first block in $SP(mod\ 5)$.
(iii) Since $SP(mod\ 13) = \{0, 1, 4, 12, 1, 9, 12, 0, 12, 9, 1, 12, 4, 1, 0, 1, 1, \dots\}$, thus we have $k_{SP}(13) = 14$ and $\alpha_{SP}(13) = 7$. Thus, $SP_7 \equiv 0(mod\ 13)$. Here $0, 1, 4, 12, 1, 9, 12$ is the first block in $SP(mod\ 13)$.

From last two illustrations, it is seen that the subscript of terms for which $SP_n \equiv 0(mod\ m)$ and $SP_{n-1} = SP_{n+1}$ contains equal number of (that is $\alpha_{SP}(m)$ number of) terms and the subscripts are in arithmetic progression with common difference $\alpha_{SP}(m)$. That is, $SP_{\alpha_{SP}(m)-1} = SP_{\alpha_{SP}(m)+1}$ and $SP_{\alpha_{SP}(m)} \equiv 0(mod\ m)$.

Thus, we can say that $SP_{\alpha_{SP}(m)u} \equiv 0(mod\ m)$, for each positive integer u. Moreover, since $SP_{k_{SP}(m)} \equiv 0(mod\ m)$, we say that $\alpha_{SP}(m)u = k_{SP}(m)$, where u is some positive integer. Thus, $\alpha_{SP}(m) \mid k_{SP}(m)$.

To illustrate this, we consider

$SP(mod\ 13) = \{0, 1, 4, 12, 1, 9, 12, 0, 12, 9, 1, 12, 4, 1, 0, 1, \dots\}$.

Then it can be seen that $SP_0 \equiv SP_7 \equiv 0(mod\ 13)$, $SP_6 = SP_8 = 12$ and $k_{SP}(13) = 14$. Thus, in this case $\alpha_{SP}(13) = 7$ and $\alpha_{SP}(13) \mid k_{SP}(13)$.

Later we will show that the value of u is always either 1 or 2. The following result gives interesting outlook about the divisibility property of suffix n.

Lemma 6. 1: $\alpha_{SP}(m) \mid n$ if and only if $m \mid SP_n$.

Proof: Let $\alpha_{SP}(m) \mid n$. Then, we have $n = n' \times \alpha_{SP}(m)$; for some $n' \in \mathbb{Z}$.

In view of the above comment, $SP_n = SP_{\alpha_{SP}(m)\times n'} \equiv 0(mod\ m)$. This gives $m \mid SP_n$.

To prove the converse part, assume that $m \mid SP_n$. Then by the definition of $\alpha_{SP}(m)$, either $\alpha_{SP}(m) = n$ or $\alpha_{SP}(m) < n$. If $\alpha_{SP}(m) = n$ then $\alpha_{SP}(m) \mid n$ is true and if $\alpha_{SP}(m) < n$ then as n lies in the simple arithmetic progression with first term 0 and common difference $\alpha_{SP}(m)$, we have $n = \alpha_{SP}(m) \times n'$. Therefore, $\alpha_{SP}(m) \mid n$ is true in any case. This completes the proof.

The following interesting divisibility property always holds for any arbitrary values of m and n.

Theorem 6. 2: $\alpha_{SP}(m) \mid \alpha_{SP}(mn)$.

Proof: By the definition of $\alpha_{SP}(m)$, we have $SF_{\alpha_{SP}(m)} \equiv 0(mod\ m)$. Therefore, $m \,\big|\, SP_{\alpha_{SP}(m)}$ is always true. Thus, $mn \,\big|\, SP_{\alpha_{SP}(mn)}$ also holds. Now for any multiple of m, $\alpha_{SP}(mn)^{\text{th}}$ position within the list of residues for $SP(mod\ m)$ will always contain zero. Thus, $m \,\big|\, SP_{\alpha_{SP}(mn)}$; that is $SF_{\alpha_{SP}(mn)} \equiv 0(mod\ m)$. Hence,

$\alpha_{SP}(m) \mid \alpha_{SP}(mn)$, as required.

To illustrate this, we consider $SP(mod\ 8) = \{0, 1, 4, 1, 0, 1, \dots\}$, In this case we observe that $\alpha_{SP}(8) = 4$. When we consider

$SP(mod\ 32) = \{0, 1, 4, 25, 16, 9, 4, 17, 0, 17, 4, 9, 16, 25, 4, 1, 0, 1\ \dots\}$,

we observe that $\alpha_{SP}(32) = 8$. Thus, $\alpha_{SP}(8) \mid \alpha_{SP}(16)$.

Definition: By $S_{SP}(m)$, we mean the first positive residue appearing after the blocks in $SP(mod\ m)$. That is $SP_{\alpha_{SP}(m)+1} \equiv S_{SP}(m)(mod\ m)$ and $S_{SP}(m)$ is the smallest such number.

Since, $SP_{\alpha_{SP}(m)} \equiv 0(mod\ m)$ and $SP_{\alpha_{SP}(m)+1} \equiv S_{SP}(m)(mod\ m)$, we have $\left(SP_{\alpha_{SP}(m)}, SP_{\alpha_{SP}(m)+1}\right) \equiv S_{SP}(m) \times (0,1)(mod\ m)$. Thus, $S_{SP}(m)$ acts like a *multiplier* of the first periodic part of $SP(mod\ m)$.

To illustrate this, we consider $m = 13$. Then since $\alpha_{SP}(13) = 7$, we have

$(SP_6, SP_7, SP_8) = (12, 0,12) \equiv 12(1, 0,1)(mod\ 13)$

Thus, $S_{SP}(13) = 12$.

Definition: $\beta_{SP}(m)$ denote the order of $S_{SP}(m)(mod\ m)$.

That is $S_{SP}(m)^{\beta_{SP}(m)} \equiv 1(mod\ m)$ and if $n < \beta_{SP}(m)$ then $S_{SP}(m)^n \not\equiv 1(mod\ m)$.

As an illustration, if we once again consider $SP(mod\ 13)$, then $S_{SP}(13) = 12$ and $12^2 \equiv 1(mod\ 13)$. Thus, $\beta_{SP}(13) = 2$.

To illustrate above definitions, we consider the following two examples:

(i) Since $SP(mod\ 3) = \{0, 1, 1, 1, 0,1, \dots\}$, clearly $k_{SP}(3) = 4$. Also, the restricted period $\alpha_P(3) = 4$ and multiplier $S_{SP-1}(3) = S_{SP+1}(3) = 1$. Thus, the order of $S_{SP}(4) = 1$ and hence $\beta_{SP}(3) = 1$.
(ii) Since $SP(mod\ 5) = \{0, 1, 4, 0, 4, 1, 0, 1, \dots\}$, then clearly $k_{SP}(5) = 6$, $\alpha_{SP}(5) = 3$ and $S_{SP-1}(m) = S_{SP-1}(m) = 4$. Since $4^2 \equiv 1(mod\ 5)$, we get $\beta_{SP}(5) = 2$.

The following resembles the theorem 4.4.3 for the sequence $\{SP_n\}$.

Theorem 6. 3: $k_{SP}(m) = \alpha_{SP}(m) \times \beta_{SP}(m)$.

Proof: Throughout the proof we consider all the congruences modulo m. Suppose that one period of $SP(mod\ m)$ is partitioned into smaller and finite subsequences $R_0, R_1, R_2, \dots, R_n, \dots$ as shown below:

$$\overbrace{0, 1, \dots, S_{SP_1}}^{R_0}, \overbrace{0, S_{SP_1}, \dots, S_{SP_2}}^{R_1}, \overbrace{0, S_{SP_2}, \dots, S_{SP_3}}^{R_2}, \dots,$$

$$\overbrace{0, S_{SP_n}, \dots, S_{SP_{n+1}}}^{R_n}, \overbrace{0, 1, \dots, S_{SP_1}}^{R_{n+1}}, \dots \qquad (21)$$

where $S_{SP_1} = S_{SP}(m)$ and every R_i $(i \geq 1)$ contains exactly one 0.

Clearly each subsequence R_i has $\alpha_{SP}(m)$ terms and $S_{SP} = S_{SP}(m)$. Also, in any R_i $(i \geq 1)$, there is exactly one zero. Hence every subsequence R_i $(i \geq 1)$ is a multiple of R_0. More precisely, we have the following congruences:

$R_1 \equiv S_{SP_1} R_0, R_2 \equiv S_{SP_2} R_0, \cdots, R_{n-1} \equiv S_{SP_{n-1}} R_0, R_n \equiv S_{SP_n} R_0.$

Now the last term in R_2 is S_{SP_3} and that of R_0 is S_{SP_1}. Also, we have $R_2 \equiv S_{SP_2} R_0$. Therefore, $S_{SP_3} \equiv S_{SP_2} \times S_{SP_1} (mod\ m)$. By the similar arguments, we have

$S_{SP_4} \equiv S_{SP_3} \times S_{SP_1}$,

$S_{SP_5} \equiv S_{SP_4} \times S_{SP_1}, \cdots$,

$S_{SP_n} \equiv S_{SP_{n-1}} \times S_{SP_1}$

Therefore, we have

$$\begin{aligned} S_{SP_n} &\equiv S_{SP_{n-1}} \times S_{SP_1} \\ &\equiv \left(S_{SP_{n-2}} \times S_{SP_1}\right) \times S_{SP_1} \\ &\equiv \left(S_{SP_{n-3}} \times S_{SP_1}\right) \times S_{SP_1} \times S_{SP_1} \\ &\quad \vdots \\ &\equiv \left(S_{SP_{n-(n-1)}} \times S_{SP_1}\right) \times \underbrace{S_{SP_1} \times \cdots \times S_{SP_1}}_{n-2 \text{ times}}. \end{aligned}$$

Therefore, $S_{SP_n} \equiv S_{SP_1}^{\,n} (mod\ m)$.

Since the order of S_{SP_1} is $\beta_{SP}(m)$, we rewrite sequence (21) as follows:

$0, 1, \ldots, S_{SP_1}, 0, S_{SP_1}, \ldots, S_{SP_1}^{\,2}, 0, S_{SP_1}^{\,2}, \ldots, 0, S_{SP_1}^{\,3}, \ldots,$

$0, S_{SP_1}^{\,\beta_{SP}(m)-1}, \ldots, 0, 1, \ldots;$

with $S_{SP_1}{}^{\beta_{SP}(m)} \equiv 1 (mod\ m)$.

Thus $\beta_{SP}(m)$ can be interpreted as the number of blocks in a single period of $SP(mod\ m)$. It now follows easily that $k_{SP}(m) = \alpha_{SP}(m) \times \beta_{SP}(m)$.

The following results will be helpful for the study of blocks within the residues of $\{SP_n\}$.

Corollary 6.4:

$SP_{n \times \alpha_{SP}(m)+r} \equiv \left(SP_{\alpha_{SP}(m)+1}\right)^n SP_r (mod\ m)$.

Proof: From above theorem, we have $R_n \equiv S_{SP_n} R_0 (mod\ m)$ and $S_{SP_n} \equiv S_{SP_1}^{\,n} (mod\ m)$. Thus, we have

$R_n \equiv S_{SP_1}^{\,n} R_0 (mod\ m)$ (22)

This shows that the r^{th} term of R_n is equal to $S_{SP_1}^{\,n}$ times the r^{th} term of R_0, when considered modulo m. Also, from the definition of $S_{SP}(m)$ we conclude that $S_{SP_1} = SP_{\alpha_{SP}(m)+1}$, when considered modulo m. Therefore, from (4.9.2) and above arguments, we can say that $SF_{n \times \alpha_{SF}(m)+r} \equiv \left(SF_{\alpha_{SF}(m)}\right)^n \times$

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$SF_r\ (mod\ m)$. This finally gives

$SF_{n\times\alpha_{SF}(m)+r} \equiv \left(S_{SF}(m)\right)^n \times SF_r\ (mod\ m)$.

Corollary 6. 5: $S_n^r \equiv S_{n\times r}(mod\ m)$.

Proof: Since $S_{SP_n} \equiv S_{SP_1}^{\ n}(mod\ m)$, we write $S_{SP_n}^{\ r} \equiv \left(S_{SP_1}^{\ n}\right)^r \equiv S_{SP_1}^{\ n\times r} \equiv S_{SP_{n\times r}}(mod\ m)$, as required

Theorem 6. 6: $\beta_{SP}(m) = 1$ or 2, for $m \geq 2$.

Proof: By Koshy [3], we have $P_n^2 = P_{n-1}P_{n+1} - (-1)^n$. Taking $n = \alpha_{SP}(m)$, we get

$SP_{\alpha_{SP}(m)} = P_{\alpha_P(m)-1}P_{\alpha_P(m)+1} - (-1)^{\alpha_{SP}(m)}$. (23)

Now, $SP_{\alpha_{SP}(m)} \equiv 0(mod\ m)$. Also, we know that

$P_{\alpha_P(m)+1} \equiv s_P(m)(mod\ m)$ and

$P_{\alpha_P(m)-1} \equiv P_{\alpha_P(m)+1}(mod\ m)$.

Therefore, by (23), we have $0 = \{s_P(m)\}\{s_P(m)\} - (-1)(mod\ m)$. Thus, we get $\{s_P(m)\}^4 \equiv 1(mod\ m)$, that is $S_{SP}(m)^2 \equiv 1(mod\ m)$. Since order of $S_{SP}(m)$ is $\beta_{SP}(m)$, we finally conclude that $\beta_{SP}(m)$ must divide 2. Hence, for any $m \geq 2$, we have $\beta_{SP}(m) = 1$ or 2.

We conclude by presenting a table displaying the values of $k_{SP}(m), \alpha_{SP}(m)$ and $\beta_{SP}(m)$ for 2 ≤ m ≤ 20.

$\boldsymbol{m}$	$\boldsymbol{k_{SP}(m)}$	$\boldsymbol{\alpha_{SP}(m)}$	$\boldsymbol{\beta_{SP}(m)}$
2	2	2	1
3	4	4	1
4	2	2	1
5	6	3	2
6	4	4	1
7	6	6	1
8	4	4	1
9	12	12	1
10	6	6	1
11	12	12	1
12	4	4	1
13	14	7	2
14	6	6	1
15	12	12	1

CONCLUSIONS

In this article, we studied the length of the novel sequence – squared Pell sequence when considered modulo 10^e. We also introduced the 'blocks' within the period of this sequence and shown that length of any one period of the squared Pell sequence always contains either 1 or 2 blocks.

Conflict of Interest:

The authors confirm that this article contents have no conflict of interest to declare for this publication.

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REFERENCES

Journal Articles:

1. Horadam, A. F. (1971): Pell identities, Fib. Quart., 9 (3), 245 – 252, 263.
2. Kramer J., Hoggatt V. E. Jr. (1972): Special cases of Fibonacci Periodicity, Fibonacci Quarterly,1 (5), 519 – 522.
3. Ömür Deveci, Erdal Karaduman (2015): The Pell Sequences in Finite Groups, Utilitas Mathematica, 96, 263 – 276.
4. Patel Rima P., Shah Devbhadra V. (2021): Blocks within the period of Lucas sequence, Ratio Mathematica, 41, 71 – 78.
5. Patel Rima P., Shah Devbhadra V. (2017): Periodicity of generalized Lucas numbers and the length of its period under modulo 2^e, *The Mathematics Today*, 33, 67 – 74.
6. Wall D. D. (1960): Fibonacci series modulo m. The American Mathematical Monthly, 67, 525 – 532.

Books:

1. Koshy Thomas (2001): Fibonacci and Lucas Numbers with Applications. John Wiley and Sons, Inc., New York.
2. Koshy Thomas (2014): Pell and Pell-Lucas Numbers with Applications, Springer Science & Business Media, New York.

Thesis:

1. Renault Marc (1996): Properties of the Fibonacci sequence under various moduli, Master's Thesis, Wake Forest University.