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Analyzing the Convergence of the Adapted General Gauss-type Proximal Point Approach for Smooth Generalized Equations

Md. Asraful Alom*, Md. Zaidur Rahman, Md. Shaharul Islam, Md. Modassir Adon

Department of Mathematics, Khulna University of Engineering & Technology, Bangladesh

***Corresponding author**

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ABSTRACT

This work analyzed the adapted general Gauss-type proximal point approach for solving smooth generalized equations like $0 \in q(x) + Q(x)$, where $Q: X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph and $q: X \to Y$ is a single-valued mapping acting between two general Banach spaces X and Y. To confirm the existence and convergence of any sequence produced by this technique under appropriate assumptions, we develop the convergence criteria of this approach by utilizing metrically regular mapping and gathered both semi-local as well as local convergence results. Lastly, we plot a numerical example to compare the semi-local convergence result of this technique.

Keywords: Metrically regular mapping, Set-valued mapping, Semi-local convergence, Fixed point lemma, Generalized Equation.

INTRODUCTION

The challenge of identifying a point $x \in \Omega \subseteq X$ also known as the solution of the problem satisfying the smooth Generalized Equation

$$0 \in q(x) + Q(x) \tag{1}$$

is the focus of this thesis, where a single-valued function $q: X \to Y$ is smooth while $Q: X \rightrightarrows 2^Y$ stands for a set-valued map with closed graph, both X and Y are Banach spaces and Ω is an open subset of X.

This dissertation deals with smooth generalized equations. Robinson first established the generalized equations as an absolute structure for a wide variety of variational issues in his works [1, 2], such as system of inequalities, system of nonlinear equations, complementary problems, equilibrium problems, variational inequalities, etc.; see in example [3-5]. Additionally, they have enough uses in applied computational sciences, economics, mathematical programming, traffic equilibrium problems, analysis of elastoplastic structures, etc.

In this research, we looked at two different convergent problems for iteratively solving generalized equations. The first of these is called semi-local convergence analysis and it focusses based on the data near the starting point x_0 , the convergence criterion is established. The second is called local convergence analysis and it concentrates on the convergence ball based on the data around a generalized equations solution. Among the most popular methods for solving the inclusion (1), the proximal point algorithm (PPA) is the one.

The PPA approach was first presented by Martinet [6] in 1970 for use in convex optimization. Rockafellar [7] examined the PPA under the general framework of monotone inclusion maximization. Numerous authors have investigated proximal point algorithm generalizations and discovered uses for this technique to solve particular variational issues in the past fifty-five years. The majority of the fast-expanding body of research on this topic has focused on various iterations of the algorithm for addressing monotone mapping-related issues, particularly monotone variational inequalities; see, for instance, [8,9]. Spingarn [10] was the first to study monotonicity in its weaker version; see Iusem *et al.* [11] for further information.

The generalized proximal point algorithm has been made available to Aragon Artacho *et al.* [12]. They have reported the local convergence analysis of the generic (PPA) for the mapping Q with set values under various metric regularity assumptions. For solving (1), Rashid *et al.* [13] gave the subsequent traditional Gauss-type proximal point algorithm (G-PPA). They obtained semi-local as well as local convergence findings in addressing problem (1). In his later work [5], Rashid created the G-PPA to solve variational inequality and produced results on both semi-local as well as local convergence.

Let O be the neighbourhood of origin and select a sequence of continuous Lipschitz functions $g_l: X \to Y$ on O with positive Lipschitz constants λ_l and $g_l(0) = 0$. Let $R(\lambda_l, x)$ stand for the subset of X for any x belongs to X and for a certain sequence of scalars λ_l which are away from *0*, that is described as bellow:

$$R(\lambda_l, x) = \{\mathfrak{d} \in X: 0 \in \lambda_l \mathfrak{d} + q(x + \mathfrak{d}) + Q(x + \mathfrak{d})\}.$$

In order to solve the generalized smooth equation (1), Dontchev and Rockafellar developed the following PPA in [3]:

Algorithm 1 (PPA)

First step: Set $l = 0$, $x_0 \in X$ and $\lambda > 0$.

Second step: Stop if $0 \in R(\lambda_l, x_l)$; Otherwise, move on to Third step

Third step: Enter $\{\lambda_l\} \subseteq (0, \lambda)$ and if $0 \notin R(\lambda_l, x_l)$, select $\mathfrak{d}_l$ such

that $\mathfrak{d}_l \in R(\lambda_l, x_l)$.

Forth step: Compose $x_{l+1} = x_l + \mathfrak{d}_l$.

Fifth step: Replace l by $l + 1$ prior to proceeding to Second step.

Keep in mind that not all of the sequences produced by Dontchev and Rockafellar [3] are convergent, and that they are not uniquely defined for a starting point close to a solution. Dontchev and Rockafellar [3] demonstrated that their algorithm produces a single, linearly convergent sequence to the solution under specific circumstances. Furthermore, it appears that the well-established approach of Alom and Rashid [14] is time-consuming. Some acceptable conditions can be utilized to avoid the sequences generated by the algorithm of Dontchev and Rockafellar [3] from all convergent. By using the proximal point strategy, they guarantee the validity of a single sequence that converges linearly to the outcome. In light of estimates using mathematics, this type of process is therefore unsuitable for mathematical applications. We look for an adapted general Gauss-type proximal point algorithm (GGPPA) in response to this obstacle. To solve the generalized equation (1) in the simplest manner, we suggest the adapted GGPPA with some novel concepts for the key theorem. We demonstrate this by substituting the metric regularity criterion for the Lipschitz-like feature.

Again, let $\mathcal{R}(g_l, x)$ denotes the subset of X and is defined by

$$\mathcal{R}(g_l, x) = \{d \in X: 0 \in g_l(d) + q(x + d) + Q(x + d)\}. \qquad (2)$$

To show that every sequence generated by the adapted GGPPA exists and to show that it converges, Alom and Rashid [14] regarded as the primary thesis that $\delta \leq min\left\{\frac{r_{\bar{x}}}{2}, \frac{\bar{r}}{2\lambda}, \frac{r_{\bar{y}}}{3\lambda}, 1\right\}$, $(\eta + 5)\kappa\lambda + \nu\kappa \leq 1$ and $\|\bar{y}\| < \lambda\delta$, during the fundamental theorem in this research, we believe that $\delta \leq min\left\{\frac{r_{\bar{x}}}{2}, \frac{3\bar{r}}{\lambda}, \frac{3r_{\bar{y}}}{4\lambda}\right\}$, $(\eta + 7)\kappa\lambda + \nu\kappa \leq 1$ and $\|\bar{y}\| < \frac{1}{3}\lambda\delta$. We show that our methodology for solving the smooth generalized equation (1) is better than the previous one. The difference between our proposed Approach 2 and Algorithm 1 is that the enhanced GGPPA generates sequences, all of which are convergent, whereas Algorithm 1 does not. Thus, the revised GGPPA that we recommend is provided below:

Algorithm 2 (Adapted GGPPA):

First step: Set $l = 0$, $x_0 \in X$ and $\lambda > 0$ and $\eta \geq 1$.

Second step: Stop if $0 \in \mathcal{R}(g_l, x_l)$; Otherwise, move on to Third step.

Third step: Enter $\{\lambda_l\} \subseteq (0, \lambda)$, $g_l(0) = 0$ and if $0 \notin \mathcal{R}(g_l, x_l)$, select d_l such that $d_l \in \mathcal{R}(g_l, x_l)$ and $\|d_l\| \leq \eta\, d(0, \mathcal{R}(g_l, x_l))$.

Forth step: Compose $x_{l+1} = x_l + d_l$.

Fifth step: Replace l by $l + 1$ prior to proceeding to Second step.

Based on Algorithm 2, we note that

(i) When $g_l(u) = \lambda_l u$, $\eta = 1$ and $\mathcal{R}(g_l, x_l)$ is singleton, Algorithm 2 becomes identical to Algorithm 1.

(ii) According to Alom *et al.* [15], the generalized Gauss-type proximal point technique is similar to Algorithm 2 if $q = 0$,

(iii) if $g_l(u) = \lambda_l u$, Algorithm 2 is equivalent to the GPPA for solving smooth generalized equation introduced by Alom and Rashid [16].

For solving (1) in the situation $q = 0$ and analyzing the results of semi-local and local convergence, Alom *et al.* [15] developed the generic version of the G-PPA. Alom *et al.* [17] established the uniformity of the GG-PPA of (1) with situation $= 0$. In order to solve the variational inequality problem, Rashid [5] developed the Gauss-type proximal point approach, which produced the result of semi-local as well as local convergence. Alom *et al.* [18] recently proposed a adapted general form of the Gauss-type proximal point algorithm (GG-PPA) to address the generalized equations in the case where $q = 0$. They also conducted an analysis of the algorithm's local and semi-local convergence properties. Also, Alom and Rahman [19] introduced the stability analysis of adapted GG-PPA for solving (1) in the case $q = 0$ using metrically regular mapping. Alom and Rashid [16] introduced the GPPA for the purpose of solving the generalized smooth equation (1) with the combinations of metrically regular mapping and Lipschitz-like continuity. Next time, Alom and Rashid [14] introduced the GGPPA in order to solve the generalized smooth equation (1) with the combinations of metrically regular mapping and Lipschitz-like continuity and obtained both the result of semi-local as well as local convergence. Alom *et al.* [20] introduced the adapted GPPA for solving the generalized smooth equation (1) and obtained the result of semi-local as well as local convergence.

To the best of our knowledge, no research has been done on semi-local analysis to solve the above generalized equations by using only metrically regular mappings which motivates us to research in this field to extend the idea. We suggest the adapted GG-PPA for resolving (1) with some changes to the vital theorem of [14] and verify this by applying metric regularity condition instead of Lipschitz-like property. We show that our approach outperforms the previous method in solving (1)

Notations and Preliminaries

Several conventional notations, basic ideas, and mathematical conclusions that will frequently be cited in the

next section is reviewed in this section of the article. It is assumed that both X and Y are general Banach spaces. The formula $Q: X \rightrightarrows 2^Y$ declare the set valued mapping from X to a subset of Y. Allow $r > 0$ and $x \in X$ a closed ball with radius r and centre at x is indicated by the symbol $B_r(x)$.

The graph of terms Q is expressed by and symbolized by $gphQ$.

$$gphQ = \{(a, b) \in X \times Y : b \in Q(a)\},$$

the domain of Q is represented by $domQ$ and is expressed by

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$domQ = \{x \in X: Q(x) \neq \emptyset\}$,

and Q^{-1} is the inverse of Q, which can be stated as

$Q^{-1}(b) = \{c \in X: b \in Q(c)\}$.

By $\|\cdot\|$, the symbol stands all the norms. Permit S and L to be, respectively, subsets of X.

$d(x, S) = inf\ \{\|x - b\|: b \in S\}$ in all cases of $x \in X$,

specifies the separation between x and S, where as

$$e(L, S) = sup\ \{d(x, S): x \in L\}$$

the excess between sets S and L is defined.

Definition 1. Inner product space: If there is a complex number (c_1, c_2) associated with every pair of vectors $(c_1, c_2) \in V$ so that the bellow properties are true, then a complex vector space as an inner product space, V is moved to:

i. $(c_1, c_2) = (\overline{c_2, c_1})$, the complex-conjugate is shown by the bar,
ii. $(c_1 x + c_2 y, w) = c_1(x, w) + c_2(y, w)$,
iii. $(c_1, c_1) \geq 0$ and $(c_1, c_1) = 0$ *iff* $c_1 = 0$.

Definition 2. Hilbert space: If a complex inner product space is complete with regard to the metric that its inner product induces, it is referred to be a Hilbert space. The metric is produced by the inner product norm.

Definition 3. Normed Linear space: A space that is linear if any vector m_1 in X has a real number associated with it, denoted by the symbol "$\|m_1\|$" (also known as the norm of m_1), then X is said to be a normed linear space.

i. $\|km_1\| = |k|\|m_1\|$, for any $m_1 \in X$ and $k \in K$,
ii. $\|m_1\| \geq 0$, and $\|m_1\| = 0$ iff $m_1 = 0$,
iii. $\|m_1 + m_2\| \leq \|m_1\| + \|m_2\|$, for any $m_2 \in X$.

Definition 4. Banach space: If under the metric generated by the norm, a normed linear space is complete as a metric space, it is referred to as a Banach space.

Parallelogram law: In a Hilbert space, if m_1 and m_2 are two vectors, then

$$||m_1 + m_2||^2 + ||m_1 - m_2||^2 = 2(\left||m_1|\right|^2 + ||m_2||^2).$$

Remark 1. The Banach space B converts to a Hilbert space if the norm of it complies with the parallelogram law and the inner product on B is expressed by

$4 < m_1, m_2 >= ||m_1 + m_2||^2 + ||m_1 - m_2||^2 + i||m_1 + im_2||^2 - i||m_1 - im_2||^2$.

We agree with the definition of mathematically regularity from [5] for mapping with set values.

Definition 5. **Metrically regular Mapping:**

Assuming $(x', y') \in gphQ$, where $Q: X \rightrightarrows 2^Y$ denotes a set-valued mapping. Let $r_{x'}$, $r_{y'}$, and κ all be greater than zero. When

$$d\big(x, Q^{-1}(y)\big) \leq \kappa\, d(y, Q(x)) \text{ for each } x \in B_{r_{x'}}(x'), y \in B_{r_{y'}}(y'), \qquad (3)$$

then at (x', y') on $B_{r_{x'}}(x')$ relative to $B_{r_{y'}}(y')$ with constant κ, one can say that the mapping Q is mathematically regular.

We revisit the ideas of Lipchitz-like continuity from [13] for set-valued mappings. Aubin first proposed this concept in [21].

Definition 6. Lipschitz-like continuity:

Let $(\bar{y}, \bar{x})$ be a member of gphγ and $\gamma: Y \rightrightarrows 2^X$ be a mapping with set-values. Assume that $r_{\bar{x}}$, $r_{\bar{y}}$ and l are all greater than 0. If the following discrepancy occurs, the mapping γ is considered Lipschitz-like for any y_1, y_2 belongs to $B_{r_{\bar{y}}}(\bar{y})$, then at $(\bar{y}, \bar{x})$ on $B_{r_{\bar{y}}}(\bar{y})$ relative to $B_{r_{\bar{x}}}(\bar{x})$ together with constant κ,

$$e\left(\gamma(y_1) \cap B_{r_{\bar{x}}}(\bar{x}), \gamma(y_2)\right) \leq \kappa \|y_1 - y_2\|. \quad (4)$$

We obtain the following lemma from [15], which proves the connection between a mapping Q of metric regularity at $(\bar{x},\bar{y})$ and the Lipschitz-like continuity of the inverse Q^{-1} at $(\bar{y}, \bar{x})$.

Lemma 1. Let $(\bar{x}, \bar{y}) \in gphQ$, where $Q: X \rightrightarrows 2^Y$ be a function with set values. Take $r_{\bar{x}}$ and $r_{\bar{y}}$ both are not less than zero. It becomes at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$ the function Q is metrically regular with constant c_1 for any $y, y' \in B_{r_{\bar{y}}}(\bar{y})$, iff at $(\bar{y}, \bar{x})$ on $B_{r_{\bar{y}}}(\bar{y})$ relative to $B_{r_{\bar{x}}}(\bar{x})$ the inverse $Q^{-1}: Y \rightrightarrows 2^X$ is Lipschitz-like with constant c_1, that is,

$$e\left(Q^{-1}(y) \cap B_{r_{\bar{x}}}(\bar{x}), Q^{-1}(y')\right) \leq c_1 \|y - y'\|. \quad (5)$$

Proof: Consider that at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$ metrically the mapping Q is regular and c_1 is constant. Consider y_1, y_2 belongs to $B_{r_{\bar{y}}}(\bar{y})$. We must demonstrate that (5) is true. To be able to demonstrate this, suppose x belongs to $Q^{-1}(y_1) \cap B_{r_{\bar{x}}}(\bar{x})$. We find

$$d\left(x, Q^{-1}(y_2)\right) \leq c_1\, d\left(y_2, Q(x)\right)$$

$$\leq c_1 \|y_1 - y_2\| \quad (6)$$

because at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$ metrically the mapping Q is regular and c_1 is constant. Therefore

$$e(Q^{-1}(y_2) \cap B_{r_{\bar{x}}}(\bar{x}), Q^{-1}(y_2)) = sup\ \{d\left(x, Q^{-1}(y_2)\right) : x \in Q^{-1}(y_1) \cap B_{r_{\bar{x}}}(\bar{x})\}$$

according to the definition of access *e*. This results in the statement that

$$e(Q^{-1}(y_2) \cap B_{r_{\bar{x}}}(\bar{x}),\ Q^{-1}(y_2)) \leq c_1 \|y_1 - y_2\|,$$

coupled with (6). This suggests that (5) is met.

Consider, however, that (5) is true. We must demonstrate that at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$, the mapping Q is metrically regular with c_1 being constant. Let x belongs to $B_{r_{\bar{x}}}(\bar{x})$ and y_2 belongs to $B_{r_{\bar{y}}}(\bar{y})$ to finish this. Given that (5) is valid for y_1 belongs to $B_{r_{\bar{y}}}(\bar{y})$, let y_1 belongs to $Q(x)$. As a result, x belongs to $Q^{-1}(y_1) \cap B_{r_{\bar{x}}}(\bar{x})$. The result

$$d\left(x, Q^{-1}(y_2)\right) \leq e\left(Q^{-1}(y_1) \cap B_{r_{\bar{x}}}(\bar{x}), Q^{-1}(y_2)\right)$$

$$\leq c_1 \|y_1 - y_2\|$$

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is then obtained from the definition of excess e. From the inequality above, if we select the minimum with regard to y_1 belongs to $Q(x)$ on both sides, we obtain

$$d\left(x, Q^{-1}(y_2)\right) \leq c_1\, d(y_2, Q(x))$$

which is true for any values of x belongs to $B_{r_{\bar{x}}}(\bar{x})$ and y_2 belongs to $B_{r_{\bar{y}}}(\bar{y})$. As a result, it can be seen that at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$, c_1 is constant and the mapping Q is metrically regular. As a result, the proof of Lemma-1 is finished.

The Lyusternik-Graves theorem was borrowed from [22]. We carry out that a set if s is bigger than 0, then z belongs to G, which is a locally closed subset of X. As a result, the set $G \cap B_s(z)$ is closed.

Lemma 2. <u>Lyusternik-Graves theorem</u>: Assuming that $(\bar{x}, \bar{y}) \in gphQ$, $Q: X \rightrightarrows 2^Y$ consideration is given to $gphQ$ being locally closed and Q being a mapping with set values. For any $\bar{y}$, let Q be metrically regular at $\bar{x}$ and have constant $\kappa > 0$. Consider a function $g : X \to Y$ that is continuous at $\bar{x}$ and with a Lipschitz constant λ such that λ is less than κ^{-1}. For $\bar{y} + g(\bar{x})$, the mapping $g + Q$ is metrically regular at $\bar{x}$ and has constant $\frac{\kappa}{1-\kappa\lambda}$.

In [23], Dontchev and Hagger established the fixed-point lemma for set-valued mappings, which generalized the fixed-point theorem from [16]. This lemma is indispensable for proving the existence of any sequence.

Lemma 3. <u>Banach fixed point Lemma:</u>

Assume that $\Psi : X \rightrightarrows 2^X$ is a mapping with predetermined values. Assume that r belongs to $(0, \infty)$, η_0 belongs to X,, and $0 < \propto < 1$ is such that

$$d(\eta_0, \Psi(\eta_0)) < r(1 - \propto) \tag{7}$$

and for all $x_1, x_2 \in B_r(\eta_0)$,

$$d(x_1, \Psi(x_2)) \leq e(\Psi(x_1) \cap B_r(\eta_0), \Psi(x_2)) \leq \propto \|x_1 - x_2\| \tag{8}$$

are each satisfied. This means that ϕ has a fixed point in $B_r(\eta_0)$, indicating that x belongs to $B_r(\eta_0)$ exists and $x \in \Psi(x)$. There is just one fixed point of Ψ in $B_r(\eta_0)$, if Ψ is also single-valued.

Convergence Analysis of the adapted GGPPA

Consider that both X and Y are general Banach spaces. Let $Q : X \rightrightarrows 2^Y$ be a mapping with set values which has locally closed graph, and let $q: X \to Y$ be a smooth function on $\Omega \subseteq X$. Let $r_{\bar{x}}$, $r_{\bar{y}}$, ν, and κ all be greater than *0* such that $\nu\kappa$ is less than *1*. We define

$$r^* = max\left\{\frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa}, \frac{2\nu r_{\bar{x}} + r_{\bar{y}}}{1-\nu\kappa}\right\} \tag{9}$$

It is evident from (9) that $r_{\bar{x}} < r^*$ and $r_{\bar{y}} < r^*$.

To prove the adapted GGPPA's semi-local convergence conclusion, we employ the following lemma.

Lemma 4. The set valued mapping $Q : X \rightrightarrows 2^Y$ should have a locally closed graph at $(\bar{x}, \bar{y})$. To define r^*, use (9). Assume that with constant κ, the mapping Q is metrically regular at $(\bar{x}, \bar{y})$ on $B_{r^*}(\bar{x}) \times B_{r^*}(\bar{y})$. With $Q(\bar{x}) = 0$, let $q: X \to Y$ be the mapping with Lipschitz continuous on $B_{r^*}(\bar{x})$ with Lipschitz constantν. Then at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\nu\kappa}$, the function $q + Q$ is metrically regular.

Proof. We find that

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$$d\big(x, Q^{-1}(y)\big) \leq \kappa\, d(y, Q(x)) \text{ for any } x \in B_{r^*}(\bar{x}),\ y \in B_{r^*}(\bar{y})$$

based on our assumption regarding ϖ. We will demonstrate that

$$d(x, (q+Q)^{-1})(y) \leq \frac{\kappa}{1-\nu\kappa} d(y, (q+Q)(x) \text{ for any } x \in B_{r_{\bar{x}}}(\bar{x}) \text{ and } y \in B_{r_{\bar{y}}}(\bar{y}).$$

For the purpose of completing this, we'll start with the induction of l and confirm that a sequence $\{x_l\} \subseteq B_{r^*}(\bar{x})$, with $x_0 = x$, like that, for $l = 0, 1, 2, ...$; meets the subsequent claims:

$$x_{l+1} \in Q^{-1}(y - Q(x_k)) \tag{10}$$

and

$$\|x_{l+1} - x_l\| \leq (\nu\kappa)^l \|x_1 - x\| \tag{11}$$

It is evident that (10) holds true when $l = 0$. Using the second condition in clause (9), we have $2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*(1-\nu\kappa)$and for $\nu\kappa < 1$, $(1 - \nu\,\kappa)$ is positive. This implies that $2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$that is, $\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$ Hence, for any x belongs to $B_{r_{\bar{x}}}(\bar{x})$ and y belongs to $B_{r_{\bar{y}}}(\bar{y},)$ we observe that

$$\big\|\big(y - q(x)\big) - \bar{y}\big\| = \|y - \bar{y} + q(\bar{x}) - q(x)\|$$

$$\leq \|q(x) - q(\bar{x})\| + \|y - \bar{y}\|$$

$$\leq \nu\|x - \bar{x}\| + \|y - \bar{y}\|$$

$$\leq \nu r_{\bar{x}} + r_{\bar{y}} \leq r^* \tag{12}$$

It becomes $y - q(x)$ belongs to $B_{r^*}(\bar{y})$. We have $x_1 \in Q^{-1}(y - q(x))$ as Q has a locally closed graph with $x_0 = x$. This demonstrates that (10) is real for $l = 0$. Additionally, we are able to write

$$\|x_1 - x\| \leq d(x, Q^{-1}(y - q(x))) \leq \kappa d(y, (q+Q)(x)) \tag{13}$$

by utilizing the Q's metric regularity condition. Moreover,

$$\|x_1 - x\| = \|x_1 - \bar{x} + \bar{x} - x\|$$

$$\leq \|x_1 - \bar{x}\| + \|\bar{x} - x\|$$

$$\leq r_{\bar{x}} + d\left(\bar{x}, Q^{-1}\big(y - q(x)\big)\right)$$

$$\leq r_{\bar{x}} + \kappa d\big(y - q(x), Q(\bar{x})\big)$$

$$\leq r_{\bar{x}} + \kappa\|y - \bar{y}\| + \kappa\|q(x) - q(\bar{x})\|$$

$$\leq r_{\bar{x}} + \kappa r_{\bar{y}} + \nu\kappa r_{\bar{x}}$$

$$= (1 + \nu\kappa) r_{\bar{x}} + \kappa r_{\bar{y}} \tag{14}$$

Therefore

$$\|x_1 - x\| \leq \|x_1 - \bar{x}\| + \|\bar{x} - x\|$$

$$\leq (1 + \nu\kappa) r_{\bar{x}} + \kappa r_{\bar{y}} + r_{\bar{x}}$$

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$$= (2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}} + r_{\bar{x}} \qquad (15)$$

As $\nu\kappa < 1$, we obtain that

$$(2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}} < \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1 - \nu\kappa} \leq r^*$$

based on the first condition in (9). Thus, we write from (15) that

$$\|x_1 - x\| \leq r^*$$

This becomes $x_1 \in B_{r^*}(\bar{x})$. By applying (15), we are able to type that

$$\|(y - q(x_1)) - \bar{y}\| = \|y - \bar{y} + q(\bar{x}) - q(x_1)\|$$

$$\leq \|y - \bar{y}\| + \|q(x_1) - q(\bar{x})\|$$

$$\leq \|y - \bar{y}\| + \nu\|x_1 - \bar{x}\|$$

$$\leq r_{\bar{y}} + \nu[(2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}]$$

$$= 2\nu r_{\bar{x}} + r_{\bar{y}} + \nu\kappa(\nu r_{\bar{x}} + r_{\bar{y}}) \qquad (16)$$

We discover from the second requirement in clause (9) that

$$2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*(1 - \nu\kappa)$$

and as $(1 - \nu\,\kappa)$ suggests a positive number less than 1 , it follows that

$2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$, that is $\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$

So, we get from (16) that

$$\|(y - q(x_1)) - \bar{y}\| \leq r^*(1 - \nu\kappa) + \nu\kappa r^* = r^*$$

It becomes $y - q(x_1) \in B_{r^*}(\bar{y})$ is true. Given that Q has a locally closed graph, it is evident that $x_2 \in Q^{-1}(y - q(x_1))$ and (10) is accurate for l equal to *1*. We can now write by using x_0 equal to x and the metric regularity assumption on Q that

$$\|x_2 - x\| \leq d\left(x, Q^{-1}(y - q(x_1))\right)$$

$$\leq \kappa d(y - q(x_1), Q(x))$$

$$\leq \kappa d(y - q(x_1), y - q(x)$$

$$\leq \nu\kappa\|x_1 - x\| \qquad (17)$$

From (14) and (17), We are able to write

$$\|x_2 - \bar{x}\| \leq \|x_2 - x\| + \|x - \bar{x}\|$$

$$\leq \nu\kappa\|x_1 - x\| + r_{\bar{x}}$$

$$\leq \nu\kappa[(1 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}] + r_{\bar{x}}$$

$$= (1 + \nu\kappa + (\nu\kappa)^2)r_{\bar{x}} + (\nu\kappa)\kappa r_{\bar{y}}$$

$$= \frac{1}{1-\nu\kappa} r_{\bar{x}} + (\nu\kappa)\kappa r_{\bar{y}} \tag{18}$$

Using $\frac{1}{1-\nu\kappa} < \frac{2}{1-\nu\kappa}$ and $\nu\kappa < \frac{1}{1-\nu\kappa}$ for any values of $\lambda\kappa$ such that $\lambda\kappa$<1 and the first condition in (9), we gain from (18) that

$$\|x_2 - \bar{x}\| < \frac{2}{1-\nu\kappa} r_{\bar{x}} + \frac{1}{1-\nu\kappa}\ \kappa r_{\bar{y}} = \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa} \leq r^*$$

This becomes $x_2 \in B_{r^*}(\bar{x})$. We are able to write by applying the metric regularity condition on Q that

$$\|x_2 - x\| \leq d(x_1, Q^{-1}(y - q(x_1))$$

$$\leq \kappa d(y - q(x_1), Q(x_1))$$

$$\leq \kappa d(y - q(x_1), y - q(x))$$

$$\leq \ \nu\kappa\|x_1 - x\|$$

$\leq \ \nu\kappa\|x_1 - x\|$. So, for $l = 1$, (11) is accurate. This demonstrates that (10), (11) holds for the built-in points $x_1, x_2, \ldots$ when $l = 0{,}1$. Suppose $x_1, x_2, \ldots \ldots x_n$ are built so that (10) and (11) are applicable to $l = 0, 1, 2, \ldots \ldots n-1$. We must make x_{n+1} sure that (10) and (11) are valid for $l = n$ by induction hypothesis. First, we'll demonstrate that x_i belongs to $B_{r^*}(\bar{x})$ for all i =1, 2,....., n. Inferring that from (11)

$$\|x_i - x\| \leq \sum_{j=0}^{i-1}\|x_{j+1} - x_j\| \leq \sum_{j=0}^{i-1}(\nu\kappa)^j\|x_1 - x\|$$

$$\leq \ \frac{1}{1-\nu\kappa}\|x_1 - x\| \tag{19}$$

In addition, we are able to write

$$\|x_i - x\| \leq \|x_i - x\| + \|x_i - \bar{x}\|$$

$$\leq \frac{1}{1-\nu\kappa}\|x_1 - x\| + \|x_i - \bar{x}\|$$

$$\leq [(1-\nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}] + r_{\bar{x}}$$

$$= \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa} \leq r^* \tag{20}$$

by using (19), (14), and the first condition in (9). From (20) for i equal to n and by applying the second circumstance in (9), it demonstrates that x_i belongs to $B_{\mathrm{r}^*}(\bar{x})$ for any $i = 1,2, \ldots \ldots n$ and

$$\|(y - q(x_n)) - \bar{y}\| \leq \|y - \bar{y}\| + \|q(\bar{x}) - q(x_n)\|$$

$$\leq \|y - \bar{y}\| + \nu\|x_n - \bar{x}\|$$

$$\leq r_{\bar{y}} + \nu\left(\frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa}\right)$$

$$= \frac{2\lambda r_{\bar{x}} + r_{\bar{y}}}{1-\nu\kappa} \leq r^*$$

Therefore, $y - q(x_n)$ belongs to $B_{r^*}(\bar{y})$. As a result, $x_{n+1} \in Q^{-1}(y - q(x_n))$ because the graph of Q is locally closed. It suggests that (10) holds true when $l = n$. Applying the metric regularity assumption on Q, we arrive at

$$\|x_{n+1} - x_n\| \leq d\left(x, Q^{-1}\big(y - q(x_n)\big)\right)$$

$$\leq \kappa d\big(y - q(x_n), Q(x_n)\big)$$

$$\leq \kappa d(y - q(x_n), y - q(x_{n-1})$$

$$\leq \kappa \|q(x_n) - q(x_{n-1})\|$$

$$\leq \nu\kappa \|x_n - x_{n-1}\| < (\nu\kappa)^n \|x_1 - x\| \qquad (21)$$

Due to the completion of the induction steps, (10) and (11) are true for all l. We discover from (21) that $x_0 = x$,

$$\|x_{n+1} - x\| \leq \textstyle\sum_{i=0}^{n} \|x_{i+1} - x_i\|$$

$$\leq \sum_{i=0}^{n} (\nu\kappa)^i \|x_1 - x\|$$

$$\leq \frac{1}{1-\nu\kappa} \|x_1 - x\| \qquad (22)$$

Applying the relation $\frac{1}{1-\nu\kappa} \|x_1 - x\| + \|x - \bar{x}\| \leq r^*$ from (20), we can determine from (22) that

$$\|x_{n+1} - \bar{x}\| \leq \|x_{n+1} - x\| + \|x - \bar{x}\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\| + \|x - \bar{x}\| \leq r^*$$

Consequently, x_{n+1} belongs to $B_{r^*}(\bar{x})$. Thus $\{x_k\}$ is a sequence of Cauchy and all of its members are in $B_{r^*}(\bar{x})$, as we can see from (21). Then, assuming the limit in (10) and the local closeness of $gphQ$ satisfying $\hat{x} \in Q^{-1}(y - q(\hat{x}))$i.e., $\hat{x} \in (q+Q)^{-1}(y)$ i.e., the sequence ends up at some $\hat{x} \in B_{r^*}(\bar{x})$, i.e., $\hat{x} = \lim_{l\to\infty} x_l$

Using (11) and (13), we discover

$$d\big(x, (q+Q)\big)^{-1}(y) \leq \|\hat{x} - x\|$$

$$= \lim_{l\to\infty} \|x_l - x\| \leq \lim_{l\to\infty} \sum_{i=0}^{l} \|x_{i+1} - x_i\|$$

$$\leq \lim_{l\to\infty} \sum_{i=0}^{l} (\nu\kappa)^i \|x_1 - x\|$$

$$\leq \frac{1}{1-\nu\kappa} \|x_1 - x\|$$

$$\leq \frac{\kappa}{1-\nu\kappa} d(y, (q+Q)(x))$$

As a result, the Lemma 4's proof is finished.

Let's say that $(q+Q)$ is metrically regular at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\nu\kappa}$ and $gph(q + Q) \cap (B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y}))$ is closed. Consider a single valued function $g: X \to Y$ with $g(0) = 0$, it has a Lipschitz

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constant λ and is Lipschitz continuous about the origin, meaning that $\kappa(\lambda+\nu)<1$. Define a mapping Q_x by

$$Q_x(\cdot)=g(\cdot-x)+q(\cdot)+Q(\cdot) \text{ for any } x\in X.$$

Then for any $s\in X$ and $y\in Y$, we obtain

$$s\in Q_x^{-1}(y)\Leftrightarrow y\in g(s-x)+q(s)+Q(s). \qquad (23)$$

In particular, $\bar{x}\in Q_{\bar{x}}^{-1}(\bar{y})$ for each $(\bar{x},\bar{y})\in gph(q+Q)$. (24)

Here $g(\cdot-\bar{x})$ is Lipschitz continuous on $O+\bar{x}$ with constant ν. Lemma 4 is therefore applied, and we assume that the mapping $Q_{\bar{x}}$ is metrically regular at $(\bar{x},\bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})\times B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(\lambda+\nu)}$. So, by Lemma 1, we say that the mapping $Q_{\bar{x}}^{-1}$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $B_{r_{\bar{y}}}(\bar{y})\times B_{r_{\bar{x}}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(\lambda+\nu)}$, that is,

$$e(Q_{\bar{x}}^{-1}(y)\cap B_{r_{\bar{x}}}(\bar{x}),Q_{\bar{x}}^{-1}(y'))\leq\frac{\kappa}{1-\kappa(\nu+\lambda)}||y-y'|| \text{ for all } y,y'\in B_{r_{\bar{y}}}(\bar{y}). \qquad (25)$$

Suppose that

$$\lim_{x\to\bar{x}} d\big(\bar{y},q(x)+Q(x)\big)=0. \qquad (26)$$

Write

$$\bar{r}=min\left\{r_{\bar{y}}-\frac{\nu r_{\bar{x}}}{2},\frac{r_{\bar{x}}(1-\kappa(\nu+2\lambda))}{4\kappa}\right\}. \qquad (27)$$

Then

$$\bar{r}>0\Leftrightarrow\lambda<\min\{\frac{2r_{\bar{y}}}{r_{\bar{x}}},\frac{1-\nu\kappa}{2\kappa}\}. \qquad (28)$$

To prove the convergence result of the adapted GGPPA, we need the following lemma. The refinement of the evidence for [35] serves as the proof.

Lemma 5. Given a constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$, let $Q_{\bar{x}}(\cdot)$ be metrically regular at $(\bar{x},\bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})\times B_{r_{\bar{y}}}(\bar{y})$ such that (27) and (28) are fulfilled. Consider $B_{r_{\bar{x}}}(0)\subseteq O$ and $x\in B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$. Then, $Q_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $B_{\bar{r}}(\bar{y})\times B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(\nu+2\lambda)}$, that is, $e(Q_x^{-1}(y_1)\cap B_{\frac{r_{\bar{x}}}{2}}(\bar{x}),Q_x^{-1}(y_2))\leq\frac{\kappa}{1-\kappa(\nu+2\lambda)}||y_1-y_2||$ for all $y_1,y_2\in B_{\bar{r}}(\bar{y})$.

In order to finish our primary conclusion, assuming a series of functions $g_l\colon X\to Y$ such that $g_l(0)=0$ are Lipschitz constants λ_l are fulfilled by Lipschitz continuity near the origin, which is identical for all l.

$$\lambda=\sup_l \lambda_l<\frac{1-\nu\kappa}{\kappa}. \qquad (29)$$

When we swap out g in (23) for g_l, we get the mapping $Q_x(\cdot)$ as follows:

$$Q_x^l(\cdot)=g_l(\cdot-x)+q(\cdot)+Q(\cdot) \text{ for each } l=0,1,2,\ldots. \qquad (30)$$

and rewrite equation (25) in the manner shown below:

$$e(Q_x^{l^{-1}}(y) \cap B_{r_{\bar{x}}}(\bar{x}), Q_x^{l^{-1}}(y')) \leq \frac{\kappa}{1-\kappa(\nu+\lambda)} ||y - y'|| \text{ for all } y, y' \in B_{r_{\bar{y}}}(\bar{y}). \quad (31)$$

Then, we have from (2) that

$$\mathcal{R}(g_l, x) = \{d \in X: 0 \in Q_x^l(x + d)\} = \left\{d \in X: x + d \in Q_x^{l^{-1}}(0)\right\}. \quad (32)$$

Again, we specify the mapping $V_x^l: X \to Y$ by

$$V_x^l(\cdot) = g_l(\cdot - \bar{x}) - g_l(\cdot - x) \text{ for each } x \in X, \quad (33)$$

Additionally, the set valued mapping $\Psi_x^l: X \rightrightarrows 2^X$ by

$$\Psi_x^l(\cdot) = \mathrm{Q}_x^{l^{-1}}[V_x^l(\cdot)]. \quad (34)$$

Thus, for each $x', x'' \in X$,

$$||V_x^l(x') - V_x^l(x'')|| = ||g_l(x' - \bar{x}) - g_l(x' - x) - g_l(x'' - \bar{x}) + g_l(x'' - x)||$$

$$\leq \left||g_l(x' - \bar{x}) - g_l(x'' - \bar{x})|\right| + \|g_l(x' - x) - g_l(x'' - x)||. \quad (35)$$

We now give the following essential result and its proof given a set of suitable conditions with initial point $\bar{x}$, arbitrary sequence produced by the GGPPA is guaranteed to exist and to converge semi-locally.

Theorem 1. Let's say that $(q + Q)$ is metrically regular at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\nu\kappa}$ and $gph(q + Q) \cap (B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y}))$ is closed. Consider a single valued function $g: X \to Y$ with $g(0) = 0$, which holds the Lipschitz continuity property around the origin with Lipschitz constant λ such that $\kappa(\lambda + \nu) < 1$. Allow $0 < \delta \leq 1$ to be such that

(a) $\delta \leq min \left\{\frac{r_{\bar{x}}}{2}, \frac{3\bar{r}}{\lambda}, \frac{3r_{\bar{y}}}{4\lambda}\right\}$,
(b) $(\eta + 7)\kappa\lambda + \nu\kappa \leq 1$,
(c) $\|\bar{y}\| < \frac{1}{3}\lambda\delta$.

Then there is some $\hat{\delta}$ is greater than zero so that one or more sequences $\{x_l\}$ are produced using Algorithm 2 and every sequence that is produced converges to a solution $x^* \in B_{\hat{\delta}}(\bar{x})$ of (1), i.e., x^* verifies that $0 \in q(x^*) + Q(x^*)$.

Proof. Consider that

$$t = \frac{\kappa}{1 - \kappa(\nu + 2\lambda)}.$$

Then by applying the assumption $(\eta + 7)\kappa\lambda + \nu\kappa \leq 1$ with $\eta > 1$, we have

$$t\eta\lambda \leq \frac{\eta\left(\frac{1 - \nu\kappa}{\eta + 7}\right)}{1 - \nu\kappa - 2\left(\frac{1 - \nu\kappa}{\eta + 7}\right)} = \frac{\eta}{\eta + 5} < 1.$$

Assumptions $\|\bar{y}\| < \frac{1}{3}\lambda\delta$ and $\lim_{x \to \bar{x}} d\big(\bar{y}, q(x) + Q(x)\big) = 0$ allow us to take $0 < \hat{\delta} \leq \delta$ such that for each $x_0 \in B_{\hat{\delta}}(\bar{x})$,

$$d\big(0, q(x_0) + Q(x_0)\big) < \frac{1}{3}\lambda\delta. \quad (36)$$

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By mathematical induction, we will then show that Algorithm 2 generates many sequences, with each sequence $\{x_l\}$ generated by Algorithm 2 satisfies the followings:

$$||x_l - \bar{x}|| \leq 2\delta, \tag{37}$$

and

$$\left||x_{l+1} - x_l|\right| \leq (t\eta\lambda)^{l+1}\delta \text{ for every } l = 0, 1, 2, \dots .... \tag{38}$$

In order to prove the inequalities (37) as well as (38), we define the constant $\hat{r}_x$ by

$$\hat{r}_x = \frac{3\kappa}{1-\kappa(\nu+\lambda)}\left(\left||\bar{y}|\right| + \lambda\left||x - \bar{x}|\right|\right) \tag{39}$$

By the assumptions $(\eta + 7)\kappa\lambda + \nu\kappa \leq 1$ and $\|\bar{y}\| < \frac{1}{3}\lambda\delta$ with $\eta > 1$, we have

$$\hat{r}_x \leq \frac{7\kappa\lambda}{1-\kappa(\nu+\lambda)}\delta \leq \frac{7}{\eta+6}\delta \leq \delta < 2\delta \text{ for every } x \in B_{2\delta}(\bar{x}). \tag{40}$$

Clearly (37) holds for $l = 0$. To prove that (38) is valid for $l = 0$, we must establish that x_1 exists i.e., $\mathcal{R}\left(g_{0,}x_0\right) \cap \hat{r}_{\bar{x}}(0) \neq \emptyset$. To do this, we will consider the mapping $\Psi_{x_0}^0$ defined by (34) and apply Lemma 3 to $\Psi_{x_0}^0$ with $\eta_0 = \bar{x}, r = \hat{r}_{x_0}$ and $\alpha = \frac{2}{3}$. It is sufficient to show that axioms (7) and (8) of Lemma 3 are valid for $\Psi_{x_0}^0$ with $\eta_0 = \bar{x}, r = \hat{r}_{x_0}$ and $\alpha = \frac{2}{3}$. By the definition of $\Psi_{x_0}^0$ in (34), we are able to write $\Psi_{x_0}^0(\bar{x}) = Q_{\bar{x}}^{0^{-1}}[V_{x_0}^0(\bar{x})]$. Therefore, we obtain

$$d\,(\bar{x}, \Psi_{x_0}^0(\bar{x})) = d(\bar{x}, Q_{\bar{x}}^{0^{-1}}[V_{x_0}^0(\bar{x})]). \tag{41}$$

Now that we know what metric regularity is, we are able to write

$$d(\bar{x}, Q_{\bar{x}}^{0^{-1}}[V_{x_0}^0(\bar{x})]) \leq \frac{\kappa}{1-\kappa(\nu+\lambda)} d\left(V_{x_0}^0(\bar{x}), Q_{\bar{x}}^0(\bar{x})\right)$$

$$= \frac{\kappa}{1-\kappa(\nu+\lambda)}||\bar{y} - V_{x_0}^0(\bar{x})|| \tag{42}$$

as $\bar{y} \in Q_{\bar{x}}^0(\bar{x})$ according to the set valued mapping definition $Q_x^l: X \rightrightarrows 2^Y$. Consequently, we derive (41) and (42)

$$d(\bar{x}, \Psi_{x_0}^0(\bar{x})) \leq \frac{\kappa}{1-\kappa(\nu+\lambda)}||\bar{y} - V_{x_0}^0(\bar{x})|| \tag{43}$$

Using the selection of λ and the concept of Lipschitz continuous mapping,

we derive from the mapping's definition $V_x^l: X \to Y$ in (33) that

$$\left||V_{x_0}^0(x) - \bar{y}|\right| = ||g_0(x - \bar{x}) - g_0(x - x_0) - \bar{y}||$$

$$\leq ||g_0(x - \bar{x}) - g_0(x - x_0)|| + ||\bar{y}||$$

$$\leq \lambda_0\left||x_0 - \bar{x}|\right| + ||\bar{y}||$$

$$= \lambda\left||x_0 - \bar{x}|\right| + ||\bar{y}||. \tag{44}$$

As $x_0 \in B_{\hat{\delta}}(\bar{x}) \subseteq B_\delta(\bar{x}) \subseteq B_{2\delta}(\bar{x})$, then by the assumption $\frac{4}{3}\lambda\delta \leq r_{\bar{y}}$ in (a) and by the assumption $||\bar{y}|| < \frac{1}{3}\lambda\delta$ in (c), we write from (44) that

$$||V_{x_0}^0(x) - \bar{y}|| \le \frac{4}{3}\lambda\delta \le r_{\bar{y}}. \quad (45)$$

This shows that for every $x \in B_{2\delta}(\bar{x})$, $V_{x_0}^0(x) \in B_{r_{\bar{y}}}(\bar{y})$. More specifically,

$$\begin{aligned} ||V_{x_0}^0(\bar{x}) - \bar{y}|| &= ||g_0(\bar{x} - \bar{x}) - g_0(\bar{x} - x_0) - \bar{y}|| \\ &\le ||g_0(0) - g_0(\bar{x} - x_0)|| + ||\bar{y}|| \\ &\le \lambda_0 \big||x_0 - \bar{x}|\big| + ||\bar{y}|| \\ &= \lambda \big||x_0 - \bar{x}|\big| + ||\bar{y}|| \quad (46) \\ &\le \frac{4}{3}\lambda\delta \le r_{\bar{y}}. \end{aligned}$$

This implies that $V_{x_0}^0(\bar{x}) \in B_{r_{\bar{y}}}(\bar{y})$. By using (42) in (43), we obtain

$$\begin{aligned} d(\bar{x}, \Psi_{x_0}^0(\bar{x})) &\le \frac{\kappa}{1-\kappa(\nu+\lambda)}(||\bar{y} - V_{x_0}^0(\bar{x})|| \\ &\le \frac{\kappa}{1-\kappa(\nu+\lambda)}(||\bar{y}|| + \lambda||(x_0 - \bar{x})||). \quad (47) \end{aligned}$$

By using (47) in (39) with $r = \hat{r}_{x_0}$ and $= \frac{2}{3}$, we get

$$d(\bar{x}, \Psi_{x_0}^0(\bar{x})) \le \left(1 - \frac{2}{3}\right)\hat{r}_{x_0} = (1-\alpha)r.$$

This demonstrates that axiom (7) of Lemma 3 is valid. Now, we demonstrate that axiom (8) of Lemma 3 is also valid. For this, consider x', x'' belongs to $B_{\hat{r}_{x_0}}(\bar{x})$. Therefore, we get $x', x'' \in B_{\hat{r}_{x_0}}(\bar{x}) \subseteq B_{2\delta}(\bar{x})$ by using (40). By the first assumption $2\delta \le r_{\bar{x}}$ in (a), We are able to write $x', x'' \in B_{\hat{r}_{x_0}}(\bar{x}) \subseteq B_{2\delta}(\bar{x}) \subseteq B_{r_{\bar{x}}}(\bar{x})$. Then from (45), we obtained that $V_{x_0}^0(x')$, $V_{x_0}^0(x'') \in B_{r_{\bar{y}}}(\bar{y})$. By using the characterization of the set-valued mapping $\Psi_x^l: X \rightrightarrows 2^X$ from (34) and using the concept of metric regularity, we can express the relationship as

$$\begin{aligned} d\left(x', \Psi_{x_0}^0(x'')\right) &= d\left(x', {Q_{\bar{x}}^0}^{-1}[V_{x_0}^0(x'')]\right) \\ &\le \frac{\kappa}{1-\kappa(\nu+\lambda)} d\big(V_{x_0}^0(x''), Q_{\bar{x}}^0(x')]\big) \\ &= \frac{\kappa}{1-\kappa(\nu+\lambda)} d\big(V_{x_0}^0(x''), V(x')]\big) \\ &= \frac{\kappa}{1-\kappa(\nu+\lambda)} ||V_{x_0}^0(x') - V_{x_0}^0(x'')||. \quad (48) \end{aligned}$$

Now, by using (35) in (48) and by the definition of Lipschitz continuous function, we observe that

$$\begin{aligned} d\left(x', \Psi_{x_0}^0(x'')\right) &\le \frac{\kappa}{1-\kappa(\nu+\lambda)}(||g_0(x' - \bar{x}) - g_0(x'' - \bar{x})|| \\ &\quad + ||g_0(x' - x_0) - g_0(x'' - x_0)||) \\ &\le \frac{2\lambda_0\kappa}{1-\kappa(\nu+\lambda)}||x' - x''|| \end{aligned}$$

$$\leq \frac{2\lambda\kappa}{1-\kappa(\nu+\lambda)}||x'-x''||. \quad (49)$$

By assumption (b), the above inequality becomes

$$d\left(x', \Psi_{x_0}^0(x'')\right) \leq \frac{2}{\eta+6}||x'-x''|| < \frac{2}{3}||x'-x''|| = \alpha||x'-x''||.$$

Thus, the axiom (8) of Lemma 3 is also valid. Given that axioms (7) and (8) of Lemma 3 are valid, We can determine that a fixed point exists $\widehat{x_1} \in B(\bar{x})$ so that $\widehat{x_1} \in \Psi_{x_0}^0(\widehat{x_1})$, that translates to $V_{x_0}^0(\widehat{x_1}) \in Q_{\bar{x}}^0(\widehat{x_1})$, that is $0 \in g_0(\widehat{x_1}-x_0) + Q(\widehat{x_1})$. This shows that $\widehat{x_1}-x_0 \in \mathcal{R}(g_{0,}x_0)$ and thus $\mathcal{R}(g_{0,}x_0) \cap \hat{r}_{\bar{x}}(0) \neq \emptyset$. Consequently, as $\eta > 1$, we can choose $d_0 \in \mathcal{R}(g_{0,}x_0)$ such that

$$||d_0|| \leq \eta\, d\left(0, \mathcal{R}(g_{0,}x_0)\right). \quad (50)$$

By Algorithm 2, $x_1 = x_0 + d_0$ is specified. Therefore, the point x_1 is generated by Algorithm 2. Additionally, from the definition of $\mathcal{R}(g_{0,}x_0)$, from (2) we are able to write

$$\mathcal{R}(g_{0,}x_0) = \{d_0 \in X: x_0 + d_0 \in Q_{x_0}^{0^{-1}}(0)\},$$

and so

$$d\left(0, \mathcal{R}(g_{0,}x_0)\right) = d\left(x_0, Q_{x_0}^{0^{-1}}(0)\right).$$

Since $Q_{\bar{x}}^l(\cdot)$ is metrically regular at $(\bar{x},\bar{y})$ on $B_{r_{\bar{x}}}(\bar{x})$ relative to $B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$, as a consequence of Lemma 5 the mapping $Q_{\bar{x}}^{k^{-1}}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $B_{\bar{r}}(\bar{y})$ relative to $B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ possessing the constant $\frac{\kappa}{1-\kappa(\nu+2\lambda)}$ for every $x \in B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$.More specifically, ${Q_{x_0}^0}^{-1}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $B_{\bar{r}}(\bar{y})$ relative to $B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa(\nu+2\lambda)}$ as the ball $B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ contains the point $\bar{x}$. Furthermore, by the assumption $\|\bar{y}\| < \frac{1}{3}\lambda\delta$ in (c) and the assumption $\delta \leq \frac{3\bar{r}}{\lambda}$ in (a), we obtain that

$$\left||\bar{y}|\right| < \frac{1}{3}\lambda\delta \leq \bar{r}.$$

It shows that $0 \in B_{\bar{r}}(\bar{y})$. According to Lemma 5, with constant $t = \frac{\kappa}{1-\kappa(\nu+2\lambda)}$, the mapping $Q_{x_0}^0(\cdot)$ is metrically regular at $(\bar{x},\bar{y})$ on $B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ relative to $B_{\bar{r}}(\bar{y})$. Therefore, using Lemma 1, we get

$$d\left(x_0, {Q_{x_0}^0}^{-1}(0)\right) \leq t\, d\left(0, Q_{x_0}^0(x_0)\right). \quad (51)$$

The equation (36) implies that

$$d\left(x_0, {Q_{x_0}^0}^{-1}(0)\right) \leq t\, d\left(0, Q_{x_0}^0(x_0)\right)$$

$$= t\, d\big(0, q(x_0) + Q(x_0)\big)$$

$$= \frac{1}{3}t\lambda\delta \leq t\lambda\delta. \quad (52)$$

As a result, we conclude that

$$d\left(0, \mathcal{R}(g_{0,}x_0)\right) = d\left(x_0,\ {Q_{x_0}^0}^{-1}(0)\right)$$

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$$\leq t\lambda\delta, \quad (53)$$

which we get from (32) and use in (52). By using (53), we obtain from Algorithm 2 that

$$||x_1 - x_0|| = ||d_0|| \leq \eta\, d\left(0, \mathcal{R}(g_0, x_0)\right)$$

$$\leq (t\eta\lambda)\delta. \quad (54)$$

It follows from this that (38) is valid for $l = 0$.

Assuming that Algorithm 2 produced the coordinates $x_1, x_2, \ldots\ldots, x_n$ we can conclude that (37), (38) hold for $l = 0,1,2,\ldots, n-1$. We demonstrate that there exists x_{n+1} such that (37), (38) are satisfied for $l = n$. The assumptions (37) and (38) are real for every $l \leq n-1$, Consequently, we derive the relationship

$$||x_n - \bar{x}|| \leq \sum_{i=0}^{n-1} ||d_i|| + ||x_0 - \bar{x}||$$

$$\leq \delta \sum_{i=0}^{n-1} (t\eta\lambda)^{i+1} + \delta$$

$$\leq \frac{t\eta\lambda}{1-t\eta\lambda}\delta + \delta \leq 2\delta, \quad (55)$$

and so $x_n \in \mathbb{B}_{2\delta}(\bar{x})$. It shows that (37) is valid for $l = n$. Using much the same reasoning as when $l = 0$, We can deduce that the mapping ${Q_{x_n}^{n}}^{-1}(\cdot)$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $B_{\tilde{r}}(\bar{y})$ relative to $B_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $t = \frac{\kappa}{1-\kappa(\nu+2\lambda)}$. Then, by using algorithm 2 once more, we have

$$||x_{n+1} - x_n|| \leq ||d_n|| \leq \eta\, d(0, \mathcal{R}^n(x_n))$$

$$\leq \eta\, d\left(x_n, {Q_{x_n}^{n}}^{-1}(y)\right) \leq \eta t\, d(0, Q_{x_n}^{n}(x_n))$$

$$\leq \eta t\, d(0, q(x_n) + Q(x_n))$$

$$\leq \eta t\, d(0, -g_{n-1}(x_n - x_{n-1}))$$

$$\leq \eta t\, ||g_{n-1}(0) - g_{n-1}(x_n - x_{n-1})||$$

$$\leq \eta t \lambda_{n-1} ||x_n - x_{n-1}||$$

$$\leq t\eta\lambda\, ||x_n - x_{n-1}|| \quad (56)$$

$$\leq t\eta\lambda (t\eta\lambda)^n \delta = (t\eta\lambda)^{n+1}\delta.$$

This demonstrates that (38) is real for $l = n$. Therefore (37), (38) are valid for every l. This suggests that $\{x_l\}$ is a Cauchy sequence which is produced by Algorithm 2 and hence there exists $x^* \in B_{r_{\bar{x}}}(\bar{x})$ such that $x_n \to x^*$. Thus, passing to the limit $x_{n+1} \in {Q_{x_n}^{n}}^{-1}(y)$ and since $gph(q+Q) \cap (B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y}))$ is closed, it follows that $0 \in q(x^*) + Q(x^*)$. Thus the proof is completed.

Imagine that $\lim_{x \to \bar{x}} d\left(0, q(x) + Q(x)\right) = 0. \quad (57)$

Theorem 1 is simplified to the subsequent consequence, it explains how the Algorithm 2 sequence locally converges, when $\bar{x}$ is a special case solution of (1), that is, $\bar{y} = 0$.

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Corollary 1 Let's say that $0 \in q(\bar{x}) + Q(\bar{x})$ is satisfied and that $\eta > 1, \lambda > 1$. Assume that $(q + Q)$ has a locally closed graph with constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$ at $(\bar{x}, 0)$ and is metrically regular there. Select a scalar sequence of length $\{\lambda_l\} \subseteq (0, \lambda)$. Then, there exists $\hat{\delta} > 0$ such that every sequence produced by Algorithm 2 with the beginning point $x_0 \in B_{\hat{\delta}}(\bar{x})$ terminates to a solution x^* satisfying that $0 \in q(x^*) + Q(x^*)$.

Proof. According to our presumption, $(q + Q)$ is metrically regular at $(\bar{x}, 0)$, where a locally closed graph with constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$ exists. Then, there are constants $r_{\bar{x}} > 0$ and $r_0 > 0$ such that with constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$, $(q + Q)$

is metrically regular at $(\bar{x}, 0)$ on $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_0}(0))$ (0), indicating that the inequality below is valid.

$d(x, (q + Q)^{-1}(y)) \leq \frac{\kappa}{1-\kappa(\nu+\lambda)} d(y, (q + Q)(x))$ for all $x \in B_{r_{\bar{x}}}(\bar{x}),\ y \in B_{r_0}(0)$.

Think of $sup_l \lambda_l = \lambda \in (0,1)$ as being such that $(\eta + 7)\kappa\lambda + \nu\kappa \leq 1$ and $x_0 \in B_{\hat{\delta}}(\bar{x})$. For every y_0 close to origin so that $gph(q + Q)$ is locally closed at (x_0, y_0), since x_0 is very close to $\bar{x}$. Allow us to take $0 < \hat{\delta} \leq \delta$ in order to achieve

$$d(0, q(x_0) + Q((x_0)) \leq \lambda\delta$$

for any $x_0 \in B_{\hat{\delta}}(\bar{x})$. Since $(q + Q)$ is metrically regular at $(\bar{x}, \bar{y})$ on $B_{r_{\bar{x}}}(\bar{x}) \times B_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa(\nu+\lambda)}$, one gets that

$$d(x, (q + Q)^{-1}(y)) \leq \frac{\kappa}{1-\kappa(\nu+\lambda)} d(y, (q + Q)(x)) \text{ for all } x \in B_{r_{\bar{x}}}(\bar{x}),\ y \in B_{r_0}(0),$$

where $0 < r_{\bar{x}} \leq \widehat{r_{\bar{x}}}$ such that $\frac{r_{\bar{x}}}{2} \leq \tilde{r}$ and $r_0 - \frac{r_{\bar{x}}\lambda}{2} > 0$. Then

$$\bar{r} = min\left\{r_0 - \frac{\lambda r_{\bar{x}}}{2}, \frac{r_{\bar{x}}(1-\kappa(\nu+2\lambda))}{4\kappa}\right\} > 0 \text{ and}$$

$$min\left\{\frac{r_{\bar{x}}}{2}, \frac{3\bar{r}}{\lambda}, \frac{3r_0}{4\lambda}\right\} > 0.$$

We can therefore select $0 < \delta \leq 1$ so that

$$\delta \leq min\left\{\frac{r_{\bar{x}}}{2}, \frac{3\bar{r}}{\lambda}, \frac{3r_0}{4\lambda}\right\}.$$

It is now common practice to check that Theorem 1's assumptions are all true. Therefore, we may finish the proof of the corollary by using Theorem 1

Numerical Test

A numerical test is provided in this part to validate the result of semi-local convergence of the adapted GGPPA

Example 1 Consider $X = Y = \mathbb{R}$, $\kappa = 0.2$, $x_0 = 0.2, \lambda = 0.1$, $\nu = 0.4$ and $\eta = 3$. Select a set-valued mapping Q on $\mathbb{R}$ by $Q(x) = \{-5x + 1, 4x + 3\}$ and a smooth function q on $\mathbb{R}$ by $q(x) = x - 1$. Also, choose a Lipchitz continuous function g_l by $g_l(x)=\frac{x}{3}$, where $g_l(0) = 0$. The mapping $(q + Q)$ with set-values is thus defined as $q(x) + Q(x) = \{-4x + 1, 5x + 2\}$ on $\mathbb{R}$. Afterward, Algorithm 2 yields a sequence that eventually meets to $x^* = 0.25$.

Take into consideration $q(x) + Q(x) = -4x + 1$. Then from the statement, it is clear that $q + Q$ is metrically regular at $(0.2, 0.2) \in gph(q + Q)$ and g_l is Lipschitz continuous in the neighbourhood of origin with Lipschitz constant $sup_l \lambda_l = \lambda = 0.1$. Then from (1), we have that

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$$\mathcal{R}(g_l, x_l) = \{d_l \in \mathbb{R} : 0 \in g_l(d_l) + q(x_l + d_l) + Q(x_l + d_l)\}$$

$$= \left\{d_l \in \mathbb{R} : \ d_l = \frac{3(1-4x_l)}{11}\right\}.$$

However, if $\mathcal{R}(g_l, x_l) \neq \emptyset$, we get that

$$0 \in g_l(x_{l+1} - x_l) + q(x_{l+1}) + Q(x_{l+1}).$$

According to this,

$$\mathrm{x}_{l+1} = \frac{3-\mathrm{x}_k}{11}.$$

As a result, we deduce from (56) that

$$||\mathrm{d}_l|| \leq \frac{\eta\kappa\lambda}{1-\kappa(\nu+2\lambda)}||\mathrm{d}_{l-1}||.$$

We can observe that for the specified values of η, κ, λ and ν, $\frac{\eta\kappa\lambda}{1-\kappa(\nu+2\lambda)} < 1$. This demonstrates that the sequence created by Algorithm 2 meets linearly, supporting the algorithm's result of semi-local convergence. The generalized equation has a solution 0.25 for $l = 4$, according to the following table 1, which was generated by the Matlab application.

x	$(q+Q)(\mathrm{x})$
0.2000	0.2000
0.2545	-0.0182
0.2496	0.0017
0.2500	-0.0002
0.2500	0.0000

Table 1: Identifying a solution

The graphic depiction of $(q+Q)(\mathrm{x})$ is shown in the following figure:

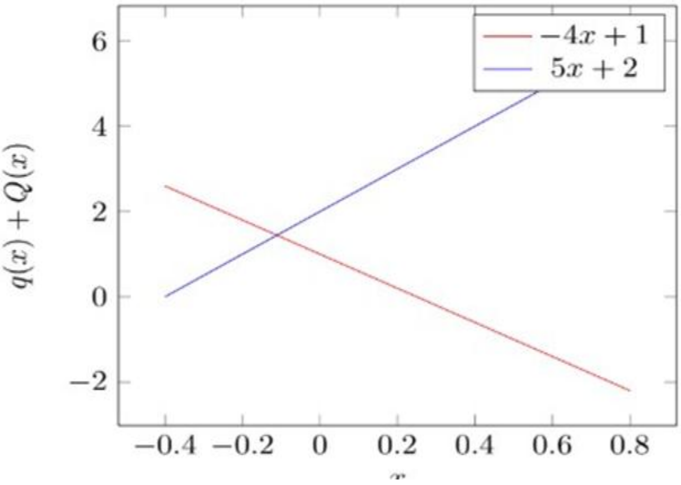

Figure 1: The graph of $(q+Q)(\mathrm{x})$.

Final Observations

The semi-local as well as local convergence findings for the adapted GGPPA specified by Algorithm 2 have been developed in this work. The generalized smooth equation (1) can be solved with the following assumptions: q is a smooth differentiable function, q is a set valued mapping that is metrically regular, q has a locally closed graph, and $g_l: X \rightarrow Y$ is a sequence of Lipschitz continuous functions such that $g_l(0) = 0$ around the origin with Lipschitz constant λ. In the event when $g_l(u) = \lambda_l(u), \eta = 1$, and $\mathcal{R}(g_l, x_l)$ is a singleton, the findings of this study align with those found in [3]. We have supported the study of semi-local convergence of the adapted GGPPA with a numerical example. The outcome builds upon and enhances the outcome found in [3, 14]. Next time, we will try to analyze the convergence of the Gauss-type proximal point method for non-smooth generalized equations.

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