# A Supercyclic Weighted Shift Has Cyclic Square

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## ABSTRACT

It is demonstrated that for a weighted abounded bilateral shift $T$ acting on $\ell_p(\mathbb{Z})$ when $1 \leq p \leq 2$ supercyclicity of $T$, weak supercyclicity of $T$, cyclicity of $T \oplus T$ and cyclicity of $T^2$ are equivalent.

A new sufficient condition for cyclicity of a weighted bilateral shift is proved, which implies, in particular, that any compact weighted bilateral shift is cyclic.

**Keywords:** Cyclicity, supercyclicity, Hypercyclicity, Quasisimilarity, Weighted bilateral shifts, Banach space, bounded linear operator.

## INTRODUCTION

All vector spaces in this article are assumed to be over the field $\mathbb{C}$ of complex numbers, $\mathbb{Z}$ is the set of whole numbers, $\mathbb{Z}_+$ the set of positive whole numbers , and $\mathbb{N}$ the set of non-negative numbers.

As is customary, symbol $L(\boldsymbol{B})$ is the space of continuous linear functions on $\boldsymbol{B}$ and $\boldsymbol{B}^*$ denotes the space of bounded linear operators on a Banach space $\boldsymbol{B}$.

For $w \in \ell_\infty(\mathbb{Z})$ and $1 \leq p \leq \infty$, $T_{w,p}$ represents the bounded linear operator on $\ell_p(\mathbb{Z})$ if $1 \leq p < \infty$ or $c_0(\mathbb{Z})$ if $p = \infty$, described using the standard framework $\{e_n\}_{n\in\mathbb{Z}}$ by

$$T_{w,p}e_n = w_n e_{n-1} \quad \text{for} \quad n \in \mathbb{Z}.$$

If furthermore $w_n \neq 0$ , $n \in \mathbb{Z}$, the weighted bilateral shift with the weight sequence $\omega$ is called for each operator $T_{w,p}$ . We have the un weighted bilateral shift in this specific instance $w_n \equiv 1$.

Remember that if there exists $x \in \boldsymbol{B}$ such that $\{T^n x : n \in \mathbb{Z}_+\}$ is dense in $B$, then a bounded linear operator $T$ on a Banach space $\boldsymbol{B}$ is said to be cyclic. $T$ is referred to as supercyclic if it has a dense $\boldsymbol{B}$ interior $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{Z}_+\}$. $T$ is likewise referred to as hypercyclic if there $x \in \boldsymbol{B}$ such that there is a dense orbit $\{T^n x : x \in \mathbb{Z}_+\}$ in $B$. Lastly, if the density is necessary in relation to the weak topology, $T$ it is referred to as weakly supercyclic or weakly hypercyclic. For further information on hypercyclicity and supercyclicity, we consult surveys [8, 9, 12]. Weakly supercyclic operators have an appealing quality in that all of their powers are cyclic and again weakly supercyclic . Ansari [2] demonstrated this result for norm supercyclicity, and the same proof holds for weak supercyclicity.

Weighted bilateral shifts cyclicity characteristics have been thoroughly researched. Salas [15, 16] described the hypercyclicity and supercyclicity of weighted bilateral shifts in terms of the weight sequences.The following simpler equivalent form of the Salas criteria is admissible, as was noted in [19, Proposition 5.1] .

**Theorem S.** For $1 \leq p \leq \infty$, a weighted bilateral shift $T = T_{w,p}$ can only be considered hypercyclic if and when any m $\in \mathbb{Z}_+$

$$\lim_{n\in\infty} \max\{\widetilde{w}(m-n+1,m), \left(\widetilde{w}(m+1,m+n)\right)^{-1}\} = 0 \tag{1.1}$$

and $T$ it is only Supercyclic if and only when any m $\in \mathbb{Z}_+$,

$$\lim_{n\in\infty} \widetilde{w}(m-n+1,m)\, \widetilde{w}(m+1,m+n)^{-1} = 0, \tag{1.2}$$

where

$$\widetilde{w}(a,b) = \prod_{j=1}^{b} |w_j| \quad \text{for } a,b \in \mathbb{Z} \text{ with } a \leq b. \tag{1.3}$$

However, it turns out that the cyclicity of a weighted bilateral shift is a far more nuanced matter, see, for example, see [10, 11, 14, 18]. It is important to note that cyclicity of a weighted bilateral shift depends on $p$, in contrast to hyper- or supercyclicity. The un weighted bilateral shift, for example, is non-cyclic on $\ell_1(\mathbb{Z})$ and cyclic on $\ell_2(\mathbb{Z})$. A weighted bilateral shift might have several sufficient and necessary criteria for cyclicity ; for example, Herrero's studies[10, 11] provide evidence of this.

One of the most important of these requirements is that a weighted bilateral shift $T$ on $\ell_p(\mathbb{Z})$ for $1 \leq p < \infty$ is non-cyclic if its adjoint has a non-empty point spectrum. The weighted bilateral shift $T_{w,p}$ with the weight sequence $w_n = a$ for $n \leq 0$ and $w_n = b$ for $n > 0$ with $0 < |b| < |a|$ is implied to be non-cyclic for any $p \in [1, \infty]$. Beauzamy's [5] initial instance of a non-cyclic weighted bilateral shift has exactly this form.

Remember that the Supercyclicity Criterion [12] states that a bounded linear operator $T$ on a Banach space $\boldsymbol{B}$ is said to satisfy the if there exist a strictly growing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers, dense subsets $E$ and $F$ of $\boldsymbol{B}$ and an a map $S : F \to F$ such that $TS_{\mathcal{Y}} = \mathcal{Y}$, for every $\mathcal{Y} \in F$, and $\|T^{n_k}x\|\,\|S^{n_k}\mathcal{Y}\| \to 0$ as $k \to \infty$ for any $x \in E$ and $y \in F$. In [12], the next two results are demonstrated.

**Theorem SC.** A supercyclic operator is one that satisfies the Supercyclicity Criterion.

**Theorem MS.** If and only if a weighted bilateral shift on $\ell_\mathrm{p}(\mathbb{Z})$ for $1 \leq \mathrm{p} < \infty$ or on $\mathrm{c}_0(\mathbb{Z})$ meets supercyclicity Criterion, it can be considered Supercyclic.

The final theorem is not that mysterious. All that is required is to consider $E = F$ the space of sequences with finite support, $S$ being the opposite of the limitation of $T$ to $F$ and utilize the theorem $S$ to identify a suitable order $\{n_k\}$. Also take note that [3, 6, 17, 19] examined the weak hypercyclicity of weighted bilateral shifts. It is demonstrated in [19] that for $p \leq 2$ any weighted bilateral shift on $\ell_p(\mathbb{Z})$, it is either weakly supercyclic or supercyclic. We expand on this duality.

**Theorem (1.1).** The following statements are equivalent, assuming $1 \leq p \leq 2$ and T be a weighted bilateral shift on $\ell_\mathrm{p}(\mathbb{Z})$ :

(C1) $T$ fulfills the Supercyclicity requirement;

(C2) $T$ is supercyclic;

(C3) $T$ is weakly supercyclic;

(C4) $T \oplus T$ is cyclic;

(C5) there is $n \geq 2$ for which $T^n$ is cyclic;

(C6) for any $n \in N$, $T^n$ is cyclic.

We emphasize that there a weakly supercyclic non-supercyclic weighted bilateral shift $T$ on $\ell_p(\mathbb{Z})$ , each $p > 2$ as demonstrated in [19] . We can observe that $(C6)$ (does not imply $(C2)$ when $p > 2$ ) since powers of a weakly supercyclic operator are cyclic. This observation and equivalency of $(C5), (C6)$ and $(C2)$ for $p \leq 2$, allow us to derive the following consequence right away.

**Corollary** $(\mathbf{1.2})$. If and only if $T^2$ is cyclic, a weighted bilateral shift $T$ occurring on $\ell_\mathrm{p}(\mathbb{Z})$ with $1 \leq \mathrm{p} \leq 2$ is supercyclic. However, there is a non-supercyclic weighted bilateral shift on $\ell_\mathrm{p}(\mathbb{z})$ for each $\mathrm{p} > 2$, with powers for all of them. It is also important to remember that weak supercyclicity of weighted bilateral shifts $T$ in [19] the necessary condition invariably results in weak supercyclicity, thus, cyclicity of $T \oplus T$. Conversely, non-supercyclic operators $T$ created in $[17, 3]$ have the non-cyclic property. In [7] a sufficient condition for a weighted bilateral shift to be unicellular (and therefore cyclic) is given. This result together with Theorem S imply that there are cyclic non-supercyclic weighted bilateral shifts on $\ell_p(\mathbb{Z})$ for $1 \leq p < \infty$ and on $c_0(\mathbb{Z})$. As a result, the requirement $n \geq 2$ in $(C5)$ is crucial. Which relationships between the criteria apply $(C1 - C6)$ for any bounded linear operator on a separable Banach space may be easily from the Theorem's (4.1.4) proof below.

Additionally, we will demonstrate that the inference $(C5) \Longrightarrow (C4)$ holds true for any weighted bilateral shift $T = T_{wT,p}$ with $1 \leq p \leq \infty$. However, general operators are not satisfied with the final implication. The Volterra operator $Vf(t) = \int_0^x f(t)\, dt$ acting on $L_2[0,1]$ , for example , [13] both satisfies $(C6)$ and does not fulfill $(C4)$.

In conclusion, we will demonstrate an additional necessary condition for the cyclicity of a weighted bilateral shift. It is not consistent with any known sufficient condition, even the most current one that Abakumov, Atzmon and Grivaux [1] have published.

**Theorem $\mathbf{A^2}$G.** Assume that $w = \{w_n\}_{n\in\mathbb{Z}}$ is a finite series of complex numbers that not zero, $\alpha_0 = 1$, $\alpha_n = (\widetilde{w}(1,n))^{-1}$ for $n > 0$ and $\alpha_n = \widetilde{w}(1+n,0)$ for $n < 0$, the definition of the numbers $\widetilde{w}(a,b)$ is found in (1.3). Additionally, suppose that $k \in \mathbb{N}$ there a sub multiplicative sequence $\{\rho_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that $ln(\rho_n) = o(\sqrt{n})$, $\alpha - n = O(n^k)$ and

$\alpha_n = O(\rho_n)$ as well $n \to +\infty$. In the event that the sequence $\{\alpha_n^{-1}\}_{n\in\mathbb{Z}}$ is not a part of $\ell_\mathrm{p}(\mathbb{Z})$, where

$\frac{1}{p} + \frac{1}{q} = 1$ , the weighted bilateral shift $T = T_{w,p}$ is cyclic .

This extremely complex outcome fails to provide a description of cyclicity for bilateral shifts that are weighted. The weight sequence criteria, for example, excide compact weighted bilateral shift. The following theorem is applicable to a greater range of weight sequences, although it becomes a weaker assertion when used with weight that mee Theorem's $A^2G$ requirements.

**Theorem** $(\mathbf{1.3})$. Let $w = \{w_n\}_{n\in\mathbb{Z}}$ be a finite series of complex numbers that are not zero, such that for any $a \in \mathbb{N}$ ,

$$\inf \{ \widetilde{w}(1,m)^{-1}\, \widetilde{w}(-j(m - a),0)^{\frac{1}{j}} : j \in \mathbb{N}, m > a\} = 0. \tag{1.4}$$

The weighted bilateral shift $T_{w,p}$ is hence cyclic for $1 \leq p \leq \infty$.

By substituting $m = a + 1$ into (1.4), we get the following corollary right away.

**Corollary (1.4).** Let $w = \{w_n\}_{n\in\mathbb{Z}}$ be a bounded series of complex numbers that are non-zero, such that

$$\lim_{n\in+\infty} \widetilde{w}(1 - n,0)^{1/n} = 0 \tag{1.5}$$

The weighted bilateral shift $T_{w,p}$ is hence cyclic for $1 \leq p \leq \infty$.

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Since $\|T^n_{w.p}\| \geq \|T^n_{w.p}e_0\| = \widetilde{w}(1-n,0)$ each quasinilpotent weighted bilateral shift for each $n \in Z_+$ implies that the spectral radius formula satisfies (5). It should be noted that a compact weighted bilateral shift is inherently quasinilpotent. Therefore, the following corollary is accurate.

**Corollary** $(\mathbf{1.5})$. A weighted bilateral shift that is quasinilpotent is cyclic. Specifically, any compact weighted bilateral shift has a cyclic structure.

If we do $j \in \mathbb{N}$ in $(4)$, the following corollary follows right away.

**Corollary** $(\mathbf{1.6})$. Let $w = \{w_n\}_{n\in\mathbb{Z}}$ be a finite series complex numbers that are not zero, for which there exists $j \in \mathbb{N}$ such that

$$\lim_{m\to+\infty} \frac{\widetilde{w}(a-jm,0)}{\left(\widetilde{w}(1,m)\right)^j} = 0 \text{ for every } a \in \mathbb{N}. \qquad (6)$$

After then, the weighted bilateral shift $T_{w,p}$ is period.

**Example** $(\mathbf{1.7})$. Assume that $a, b > 0,\ 0 < \alpha \leq 1$ and $w = \{w_n\}_{n\in\mathbb{Z}}$ is a series of positive numbers such that $1 - w_n \sim b(-n)^{-\alpha}$ as $n \to -\infty$. It is from corollary (4.1.10) that the weighted bilateral shift $T_{\omega,p}$ is cyclic for $1 \leq p \leq \infty$. Conversely, Theorem $A^2G$ is only relevant if $\alpha > \frac{1}{2}$. Also take note that by Theorem $S$, (4.1.1). $T$ is non-supercyclic if $b < a$ and supercyclic if $b > a$ .

**Proof of Theorem** $(\mathbf{1.1})$. We begin with these three simple, well- known, yet elegant observations.

**Lemma** $(\mathbf{2.1})$. Allow $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be Banach spaces, and $T_1 \in L(\boldsymbol{B}_1)$, $T_2 \in L(\boldsymbol{B}_2)$ such that satisfies the existence of a bounded linear operator $J: \boldsymbol{B}_1 \to \boldsymbol{B}_2$ with dense range $T_2J = JT_1$. Consequently, cyclicity of $T_1$ implies cyclicity of $T_2$.

**Proof.** Remember that span$\{T_2^nJx: n \in \mathbb{Z}_+\} = J$(span $\{T_1^n\, x : n \in \mathbb{Z}_+\}$) for every $x \in \boldsymbol{B}_1$. Thus, for every cyclic vector $x$ for $T_1$ , there is a cyclic vector for $T_2$.

**Remark .** Lemma (2.1) holds true even when cyclicity is substituted with hypercyclicity, supercyclicity, weak hypercyclicity or weak supercyclicity, as demonstrated by the same argument.

**Lemma** $(\mathbf{2.2})$. Assuming that $\boldsymbol{B}$ is a Banach space, and $T \in L(\boldsymbol{B})$, the operator $T \oplus T^*$, acting on $\boldsymbol{B} \times \boldsymbol{B}^*$ it is non-cyclic.

**Proof.** Let $(x, f) \in \boldsymbol{B} \times \boldsymbol{B}^*$ be distinct from zero. Consequently, the non-zero continuous linear functional $F$ on $\boldsymbol{B} \times \boldsymbol{B}^*$ is defined by $F(\mathcal{Y}, g) = f(\mathcal{Y}) - g(x)$. We've got,

$$F(T^n x, T^{*n} f) = f(T^n x) - T^{*n} f(x) = f(T^n x) - f(T^n x) = 0 \text{ for any } n \in \mathbb{Z}_+.$$

Therefore , the kernel of a non-zero continuous linear functional contains orbit of any non-zero vector under $T \oplus T^*$. Consequently , $T \oplus T^*$ is not cyclic.

**Corollary** $(\mathbf{2.3})$. Suppose that $\boldsymbol{B}$ is a Banach space and $T \in L(\boldsymbol{B})$ that there is a bounded linear operator $J$ : $\boldsymbol{B} \to \boldsymbol{B}^*$ with dense range satisfies $T^*J = JT$. In that case, the operator $T \oplus T$ acting on $\boldsymbol{B} \oplus \boldsymbol{B}$ is not cyclical.

**Proof.** Given that $T^*J = JT$, we have $(T \oplus T^*)(I \oplus J) = (I \oplus J)(T \oplus T)$. Let's say that $T \oplus T$ is cyclic.

Lemma (2.1) suggests that $T \oplus T^*$ is cyclic since $I \oplus J: \boldsymbol{B} \times \boldsymbol{B} \to \boldsymbol{B} \times \boldsymbol{B}^*$ is bounded and has dense range, , which is not feasible based on Lemma (2.2).

**Lemma $(4.1.16)$.** On a Banach space $\boldsymbol{B}$, let $j \in \mathbb{N}$ and $T$ be bounded linear operator with dense range that $T^j$ is cyclic. Additionally, let $z = e^{2\pi i/j}$. Next, the operator functioning

$$S = T \oplus zT \oplus z^2T \oplus \ldots \oplus z^{j-1}T,$$

on $\boldsymbol{B}^j$ is cyclic.

**Proof.** For $T^j$, Let $x$ be a cyclic vector. The space of polynomials on one variable with complex coefficients is $\mathcal{P} = \mathbb{C}[t]$, then $L = \{r(T^j)x : r \in \mathcal{P}\}$ dense in $\boldsymbol{B}$. The spaces $T(L), \ldots, T^{j-1}(L)$ are crowded in $\boldsymbol{B}$ since $T$ has a dense range. Verifying that $u = (x, x, \ldots, x) \in \boldsymbol{B}^j$ is a cyclic vector is sufficient for $S$. Let $M$ be the orbit's closed linear span under $S, 0 \leq k \leq j-1$ and $r \in \mathcal{P}$. Then

$$S^k r(S^j)u \ = \ \left(T^k r(T^j)x, z^k T^k r(T^j)x, \ldots, z^{k(j-1)} T^k r(T^j)x\right) \in M.$$

Thus, $M$ includes the vectors of the shape $\left(a, z^k a, \ldots, T^j z^{k(j-1)a}\right)$ for $a \in T^k(L)$ and $0 \leq k \leq j-1$.

Since $M$ is closed and $T^k(L)$ dense in $\boldsymbol{B}$, we may conclude that

$$M \supseteq N_k = \left\{\left(a, z^k a, \ldots, z^{k(j-1)a}\right) : a \in \boldsymbol{B}\right\} \text{ for } 0 \leq k \leq j-1.$$

Finally, the matrix $\{z^{kl}\}_{k,l=0}^{j-1}$ is invertible since its determinant is a Vander Monde type. The latter matrix is invertible, which indicates that the union of $N_k$ for $0 \leq k \leq j - 1$ spans $\boldsymbol{B}^j$. As a result $M = \boldsymbol{B}^j$, $u$ is a cyclic vector for $S$.

For weighted bilateral shifts, the final lemma can be phrased more elegantly. Remember, if $|w_n| = |u_n|$ for any $n \in \mathbb{Z}$, the weighted bilateral shifts $T_{w,p}$ and $T_{u,p}$ then are isometrically similar for each $p \in [1, \infty]$. Indeed, take the sequence $\{d_n\}_{n \in \mathbb{Z}_+}$ defined as $d_0 = 1$, $d_n = \widetilde{w}(1,n)/\tilde{u}(1,n)$.

For $n \geqslant 1$ and $d_n = \tilde{u}(n+1,0)/\,\widetilde{w}(n+1,0)$ for $n < 0$. Then $|d_n| = 1$ for each $n \in \mathbb{Z}_+$ and thus the diagonal operator $D$, it acts on the basic vectors using the formula $De_n = d_n e_n$ for $n \in \mathbb{Z}_+$, an invertible isometry. That is easy to check $T_{w,p} = D^{-1} T_{u,p} D$. That is, $T_{w,p}$ and $T_{u,p}$ are isometrically comparable. Any $T_{w,p}$ is particularly similar to $zT_{w,p}$ if $z \in \mathbb{C}$ and $|z| = 1$. This observation together with the previous lemma, leads to the following consequence.

**Corollary $(2.5)$.** Let us consider $T = T_{w,p}$ a weighted bilateral shift that $T^j$ is cyclic. Then $T \oplus T$ is cyclical.

**Proof.** According to lemma (2.4), the operator $T \oplus zT \oplus \ldots \oplus z^{j-1}T$ is cyclic, where $z = e^{2\pi i/j}$. Based on the preceding observation, it follows that $T$ is similar to $z^k T$ for $0 \leq k \leq j-1$. As a result, the direct sum of $j$ copies of $T$ is cyclic, implying that $T \oplus T$ is cyclic.

The following lemma establishes a sufficient condition for a direct sum of two weighted bilateral shifts to be non-cyclic.

**Lemma $(2.6)$**. Consider $w$ a bounded succession of non-zero complex numbers, $p_1, p_2 \in [1, \infty]$ and

$$q = q(p_1, p_2) = \begin{cases} \dfrac{p_1 p_2}{p_1 p_2 - p_1 - p_2} & \text{if } p_1 + p_2 < p_1 p_2\,, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Assume there exists $m \in \mathbb{Z}_+$ such that $a = \{a_n\}_{n \in \mathbb{Z}_+} \in \ell_q$, where

$$a_n = \frac{\widetilde{w}(m+1, m+n)}{\widetilde{w}(m-n+1, m)} \quad \text{for } n \in \mathbb{Z}_+. \tag{2.2}$$

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Then $T_{w,p_1} \oplus T_{w,p_2}$ is non-cyclical.

**Proof.** For brevity $\boldsymbol{B}_p = \ell_p(\mathbb{Z})$ if $1 \le p < \infty$ and $\boldsymbol{B}_\infty = c_0(\mathbb{Z})$. Consider the bilateral sequence $\{d_n\}_{n\in\mathbb{Z}}$ described by

$d_0 = 1,\ d_n = \prod_{j=1}^{n} \frac{w_j}{w_{2m+1-j}}$ if $n > 0$ and $d_n = \prod_{j=1}^{|n|} \frac{w_{2m+j}}{w_{1-j}}$ if $n < 0$.

It is easy to confirm that $d_{n+m} = d_{m-n} = (\widetilde{w}(m-n+1,m))^{-1}\, \widetilde{w}(1,m) a_n$ for each $n > m$.

Since then $a \in \ell_q$, we have $d \in \ell_q(\mathbb{Z})$. Let $p_1' \in [1,\infty]$ be defined by the formula $\frac{1}{p_1} + \frac{1}{p_1'} = 1$. Based on the definition of $q$, we have $\frac{1}{p_1'} \le \frac{1}{q} + \frac{1}{p_2}$.

The Hölder inequality allows us to define a bounded linear operator $J : \boldsymbol{B}p_2 \to \boldsymbol{B}p_1'$ on the canonical basis $Je_n = d_n e_{2m-n}$. It is simple to prove, by computing the values of the operators on the basic vectors $(e_k, e_n)$, that $(T_{\omega,p_1} \oplus S)(I \oplus J) = (I \oplus J)(T_{w,p_1} \oplus T_{w,p_2})$, where $S$ is the bounded linear operator $\boldsymbol{B}_{p_1'}$ defined as $Se_n = w_{n+1} e_{n+1}$ for $n \in \mathbb{Z}$.

Assume $T_{w,p_1} \oplus T_{w,p2}$ it is cyclical. Since $I \oplus J$ has dense range, Lemma (2.1) implies that $T_{w,p_1} \oplus S$ is cyclic, this is impossible, according to Lemma (2.2), if $1 < p_1 \le \infty$, then and if $1 \le p_1 < \infty$, then $T_{w,p_1} = S^*$. In any case, $T_{w,p_1} \oplus S$ is the direct sum of one operator and its dual.

The following corollary is the special case $p_1 = p_2$ of the preceding lemma.

**Corollary (2.7).** Consider w a bounded sequence of non-zero complex numbers, $p$ and $q = \infty$ if $p \le 2$, $q = p/(p-2)$, if $p > 2$. Assume that there exists $m \in \mathbb{Z}_+$ such that $a = \{a_n\}_{n\in\mathbb{Z}_+} \in \ell_q$, as stated in (8). Then $T_{w,p} \oplus T_{\omega,p}$ is not cyclic.

To show the next proposition, we use Lemma (4.1.18) in the instance

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

**Proposition (2.8).** Consider $w = \{w_n\}_{n\in\mathbb{Z}}$ a bounded series of non zero complex numbers. Then $T_{w,p}$ is supercyclic if and only if $T_{w,p} \oplus T_{w,p'}$ it is cyclic, where $\frac{1}{p} + \frac{1}{p'} = 1$.

**Proof.** According to theorem $S$, supercyclicity of $T_{w,p}$ does not depend on $p$. Specifically, $T_{w,p}$ is supercyclic if and only if $T_{w,p'}{}'$ is supercyclic. So, without losing generality, we can assume that $p \le p'$. If $T_{w,p}$ is not supercyclic, $T_{w,p}$ it satisfies the supercyclicity Criterion, according to the theorem $MS$. Since an operator $T$ satisfies the supercyclicity Criterion if and only if $T \oplus T$ it does, we have that $T_{w,p} \oplus T_{w,p}$ fulfills the supercyclicity Criterion and $T_{w,p} \oplus T_{w,p}$ is hence cyclic. Since $\ell_p(\mathbb{Z}) \times \ell_p(\mathbb{Z})$ is tightly constantly in $\ell_p(\mathbb{Z}) \times \ell_{p'}(\mathbb{Z})$ if $p' < \infty$, and into $\ell_p(\mathbb{Z}) \times c_0(\mathbb{Z})$ if $p' = \infty$, we can see that cyclicity $T_{w,p} \oplus T_{w,p}$ entails cyclicity $T_{w,p} \oplus T_{w,p'}$. Thus, $T_{w,p} \oplus T_{w,p'}$ it is cyclic.

Assume that $T_{w,p}$ is not supercyclic. The presence of $m \in \mathbb{Z}_+$ such that (1.2) is not satisfied is implied by the theorem $S$. Then $a = \{a_n\}_{n\in\mathbb{Z}_+} \in \ell_\infty$ where $a$ is defined as in (2.2). It is easy to observe (1.2) that

$q = q(p,p') = \infty$. By Lemma (4.1.18), $T_{w,p} \oplus T_{w,p'}$ is not cyclic.

**Proposition (2.9).** Let us consider $T$ a bounded linear operator on a separable Banach space $\boldsymbol{B}$. Then the conditions (C1– C6) of Theorem (1.1), are connected in the following manner:

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$$(C4) \Leftarrow (C1) \Rightarrow (C2) \Rightarrow (C3) \Rightarrow (C6) \Rightarrow (C5).$$

**Proof.** By Theorem $SC$ (4.1.2), $(C1)$ implies $(C2)$. The implication $(C1) \Rightarrow (C4)$ follows from the same theorem and the fact that $T$ satisfies the supercyclicity Criterion if and only if $T \oplus T$ does. Since the powers of a weakly supercyclic operator are weakly supercyclic, we see that $(C3)$ implies $(C6)$. The implications

$(C2) \Rightarrow (C3)$ and $(C6) \Rightarrow (C5)$ are trivial.

**Proof of Theorem $(\mathbf{4.1.4})$.** According to Corollary (2.5) $(C5)$ implies $(C4)$. By Theorem $MS$(4.1.3), $(C2)$ implies $(C1)$. Taking into account Proposition $(\mathbf{2.9})$, we see that it suffices to show that $(C4)$ implies $(C2)$.

If $T \oplus T$ is cyclic on $\ell_p(\mathbb{Z}) \oplus \ell_p(\mathbb{Z})$ then, since $p \le 2$, it is cyclic on $\ell_p(\mathbb{Z}) \oplus \ell_q(\mathbb{Z})$. By Proposition (4.1.20), $T$ is supercyclic, which $(C4)$ implies $(C2)$.

**Proof of Theorem (1.3).** Refer to [8, p. 348–349] for general finding on universal families.. Let us consider $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ a family of continuous maps from a complete metric space $X$ to a separable metric space. If and only if the set $\{(x, f_\alpha(x)) : x \in X, a \in A\}$ is dense in $X \times Y$ , then the set $\{x \in X : \{f_\alpha(x) : a \in A\}$ is dense in $Y\}$ of universal elements for $\mathcal{F}$ is dense in $X$ .

The following theorem can be obtained by directly applying this result to the family

$\{r(T) : r \in \mathcal{P}\}$, where $T$ is a bounded linear operator on a Banach space.

**Theorem DC .** Define $\boldsymbol{B}$ a bounded linear operator and a separable Banach space, respectively. If and only if the set $\{(x, r(T)x) : x \in \boldsymbol{B}, r \in \mathcal{P}\}$ is dense in $\boldsymbol{B} \times \boldsymbol{B}$, then the set of cyclic vectors for $T$ is dense in $\boldsymbol{B}$.

When a bounded linear operator $T$ acts on a subset of a Banach space $\boldsymbol{B}$ and is dense in $\boldsymbol{B}$, we say that the subset $\boldsymbol{A}$ if $\bigcup_{r \in \mathcal{P}} r(T)(A)$ is cyclic. The following refinement is allowed by Theorem $DC$.

**Corollary (3.1).** Let $T : \boldsymbol{B} \to \boldsymbol{B}$ be a bounded linear operator, $\boldsymbol{B}$ a separable Banach space, and $A, B$ two cyclic subsets of $T$. Additionally, assume that the dual operator's $T^*$ point spectrum $\sigma_p(T^*)$ has an empty interior. Then the set of cyclic vectors for $T$ is thus dense in $\boldsymbol{B}$ if and only if $x \in A$, $\mathcal{Y} \in B$ and $\varepsilon > 0$, there exist $u \in \boldsymbol{B}$ and $r \in \mathcal{P}$ such that for any $\|x - u\| < \varepsilon$ and $\|y - r(T)u\| < \varepsilon$.

**Proof.** The part that states 'only if' follows directly from Theorem DC. The 'if' portion still needs to be proven. Evidently

$$B \subseteq M_\delta = \overline{\bigcup_{r \in \mathcal{P}, x \in A} r(T)(x + \delta U)} \text{ for any } \delta > 0, \text{ where } U = \{x \in \boldsymbol{B} : \|x\| < 1\}.$$

As a result, for any $\delta > 0$ and $q \in \mathcal{P}$, we have $q(T)(M_\delta) \subseteq M_\delta$ and for that reson $q(T)(B) \subseteq M_\delta$. Given that every $M_\delta$ is closed and $\boldsymbol{B}$ is cyclic , we have $M_\delta = \boldsymbol{B}$ for every $\delta > 0$. Let $\mathcal{P}^\dagger$ be the collection of polynomials $q \in \mathcal{P}$ with all of zeros in $\mathcal{C} \setminus \sigma_p(T^*)$. Then $q(T)$ provides a wide range for any $q \in \mathcal{P}^\dagger$ Specifically, we observe that the set

$$q(T)\left(\bigcup_{r \in \mathcal{P}, x \in A} r(T)(x + \delta U)\right) = \bigcup_{r \in \mathcal{P}, x \in A} r(T)\,(q(T)x + \delta q(T)(U))$$

is thick $\boldsymbol{B}$ for all purposes $\delta > 0$ and $q \in \mathcal{P}^\dagger$. Lastly, given that $q(T)(U) \subseteq \|q(T)\| U$ our data

$\bigcup_{r \in \mathcal{P}, x \in q(T)(A)} r(T)(x + \varepsilon U)$ is dense $\boldsymbol{B}$ for every $\varepsilon > 0$ and $q \in \mathcal{P}^\dagger$.

Using the concept of $\mathcal{P}^\dagger$and cyclicity of $A$ for $T$ , we obtain $\bigcup_{q \in \mathcal{P}^\dagger} q(T)(A)$ Since $\sigma p(T^*)$ has empty interior.

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The final two visualizations suggest that the collection $\{(x, r(T)x) : x \in \boldsymbol{B}, r \in \mathcal{P}\}$ is dense in $\boldsymbol{B} \times \boldsymbol{B}$.

There is density in $\boldsymbol{B}$ the set of cyclic vectors for $T$ by Theorem DC.

We'll use the corollary mentioned above for weighted bilateral shifts. The sentence that follows is an example of a corollary (3.1).

**Corollary (3.2).** Let $\{f_j\}_{j\in\mathbb{Z}}$ be a sequence of elements of $\boldsymbol{B}$ such that $\{f_j : j \in \mathbb{Z}\}$ is dense in $\boldsymbol{B}$ and

$Tf_j = f_j + 1$ for each $n \in \mathbb{Z}$, and let $T$ be a bounded linear operator on a Banach space. If and only if there exists $r \in \mathcal{P}$ and $u \in \boldsymbol{B}$ such that $\|f_{-k} - u\| \le \varepsilon$ and $\|f_{-n} - r(T)u\| \le \varepsilon$ and for any $n, k \in \mathbb{N}$, $n > k$ and any $\varepsilon > 0$, then the set of cyclic vectors for $T$ is dense .

**Proof.** The condition 'only if' is a superfluous outcome of Theorem DC. The 'if' portion still needs to be proven. Let $A = \{f_m : m < 0\}$. Given that $Tf_j = f_{j+1}$ each $n \in \mathbb{Z}$ of the things we have

$\bigcup_{r\in\mathcal{P}} r(T)(A) = \text{span}\,\{f_j : j \in \mathbb{Z}\}$ is dense in $\boldsymbol{B}$.

Thus, $A$ a cyclic set is $T$. Allow $x, \mathcal{Y} \in A$. Then $x = e_{-k}$ and $\mathcal{Y} = e_{-n}$ in some cases $k, n \in \mathbb{N}$. If $n \le k$, then a constant $c \in \mathbb{C}$ such that $r(T)x = \mathcal{Y}$, where $r(\mathcal{Z}) = c\mathcal{Z}^{k-n}$ exists. Specifically, $\|x - u\| = 0 < \varepsilon$ and $\|\mathcal{Y} - r(T)u\| = 0 < \varepsilon$ for $u = x$ regarding any $\varepsilon > 0$. If $n > k$ and $\varepsilon > 0$ when the presumptions are met, there is $r \in \mathcal{P}$ and $u \in \boldsymbol{B}$ so that $\|x - u\| < \varepsilon$ and $\|\mathcal{Y} - r(T)u\| < \varepsilon$. There is still corollary (3.1) to apply.

**Proposition (3.3).** Let $\{f_j\}_j \in \mathbb{Z}$ be a sequence of elements of $\boldsymbol{B}$ such that $\text{span}\{f_j : j \in \mathbb{Z}\}$ is dense in $\boldsymbol{B}$, and let $T$ be a bounded linear operator on a Banach space $\boldsymbol{B}$ and $Tf_j = f_{j+1}$ for each $j \in \mathbb{Z}$.

Moreover, suppose that

$$\inf\left\{\|f_{-m}\|\left\|f_{(m-a)j}\right\|^{1/j} : j \in \mathbb{N}\ \ m \geqslant a\right\} = 0 \text{ for any } a \in \mathbb{N}. \tag{9}$$

Then $T$ contains a dense collection of cyclic vectors.

**Proof.** Suppose $\varepsilon > 0$ and $n, k \in \mathbb{N}$ are such that $n > k$. We examine $x_m \in \boldsymbol{B}$ and discuss a polynomial $q_{j,m}$ defined by for any $j \in \mathbb{N}$ and $m \geqslant n > k$ ,

$$x_m = f_{-k} - \frac{\varepsilon}{\|f_{-m}\|} f_{-m} \text{ and } q_{j,m}(\mathcal{Z}) = -\frac{\|f_{-m}\|\mathcal{Z}^{m-n}}{\varepsilon}\sum_{i=0}^{j-1}\left(\frac{\|f_{-m}\|\,\mathcal{Z}^{m-k}}{\varepsilon}\right)^{l}.$$

It is simple to verify that $q_{j,m}(T)x_m = f_{-n} - (\|f_{-m}\|/\varepsilon)^j f_{(m-k)j-n}$ by using the fact that $Tf_l = f_{l+1}$ for each $l \in \mathbb{Z}$ and the summation formula for a finite geometric progression. Therefore

$$\left\|q_{j,m}(T)x_m - f_{-n}\right\| = \left\|\,(\|f_{-m}\|/\varepsilon)^j f_{(m-k)j-n}\right\|.$$

Assume $a = n + k$ for the moment that $m \geqslant a$. Next

$$\left\|\,f_{(m-k)j-n}\right\| = \left\|\,f_{(m-a)j+n(j-1)k}\right\| = \left\|T^{n(j-1)} f_{(m-a)j}\right\| \le \|T^n\|^{j-1}\left\|\,f_{(m-a)j}\right\|.$$

The last two shows provide us with

$$\left\|q_{j,m}(T)x_m - f_{-n}\right\| \le (\|f_{-m}\|/\varepsilon)^j \|T^n\|^{j-1}\left\|\,f_{(m-a)j}\right\|.$$

Therefore, it can be determined that $j \in \mathbb{N}$ and $m \geqslant a$ such that the inequality above's right hand side does not exceed $\varepsilon$. In this instance $\left\|q_{j,m}(T)x_m - f_{-n}\right\| \le \varepsilon$. Given that Corollary (3.2) implies that $T$ possesses a dense set of cyclic vectors, it follows that $\|x_m - f_{-k}\| = \varepsilon$ from the definition of $x_m$.

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**Proof of Theorem (1.3).** We have previously stated that $|\omega_n| = |u_n|$, for any $n \in \mathbb{Z}$, the weighted bilateral shifts $T_{\omega,p}$ and $T_{u,p}$ are isometrically comparable for each $p[1,\infty]$. Therefore, we can presume, without losing generality, that $\omega_n > 0$ for each $n \in \mathbb{Z}$.

Allow $f_n = c_n e_{-n}$ for $n \in \mathbb{Z}$, where $c_n = 1$ if $n = 0$, $c_n = \widetilde{\omega}(1-n,0)$ if $n > 0$ and $c_n = (\widetilde{\omega}(1,-n))^{-1}$ if $n < 0$. Seeing that $Tf_n = f_{n+1}$ for each $n \in \mathbb{Z}$. Clearly span{ $f_n : n \in \mathbb{Z}$} = span { $e_n : n \in \mathbb{Z}$} is dense is simple. Since is now $\| f_n\| = c_n$, (9) is equal to (4), it is still necessary to apply Proposition (3.3).

**Final thoughts**

Another dichotomy for weighted bilateral shifts is presented in [20], which is also worth mentioning. In particular, if there $T$ is a weighted bilateral shift on $\ell_p(\mathbb{Z})$ with $1 < p < \infty$ ,then it is either $T$ , $T^n x/\|T^n x\|$ weakly convergent to zero or Supercyclic as $n \to \infty$ for each non-zero $x \in \ell_p(\mathbb{Z})$. Similarly, weighted bilateral shifts on $c_0(\mathbb{Z})$ succeed and weighted bilateral shifts on $\ell_1(\mathbb{Z})$ fail. We want to emphasize that the following issue is yet unresolved.

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