On Generalized Jordan Derivations and Centralizers in Torsion-Free Semiprime Rings: Structural Characterizations and Equivalences

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ABSTRACT

This paper investigates the structure of generalized Jordan derivations and generalized Jordan centralizers on torsion-free semiprime rings. We establish conditions under which every nonzero generalized Jordan derivation is a derivation mapping the ring into itself. Similarly, we prove that every generalized Jordan centralizer coincides with a two-sided centralizer. The approach builds upon several supporting lemmas to demonstrate that torsion-free restrictions and semiprimeness provide sufficient conditions for such equivalences. These results extend classical findings on derivations and centralizers in associative ring theory and contribute to the structural analysis of operator mappings in algebraic systems.

Keywords: Semiprime rings; Jordan derivations; generalized Jordan derivations; centralizers; torsion-free rings; ring theory; algebraic structures.

INTRODUCTION

The study of operator mappings such as derivations, Jordan derivations, and centralizers has been central to the development of modern ring theory and its applications [1]. Derivations, which generalize the concept of differentiation into algebraic structures, serve as powerful tools in detecting symmetries and measuring noncommutativity within rings [2]. Likewise, centralizers, which capture the extent to which elements commute with operators, provide valuable insights into the internal consistency and commutative behavior of algebraic systems [3]. These mappings have been extensively investigated in prime and semiprime rings, with rich connections to operator algebras, functional identities, and commutativity-preserving transformations [4, 5, 6, 7, 8].

Over the years, mathematicians have introduced *generalized versions* of these operators to extend the scope of study beyond classical derivations and centralizers [9]. A generalized Jordan derivation associates an additive mapping with a Jordan derivation, thereby enlarging the class of admissible operators and providing a more flexible framework for analyzing ring structures [10]. Similarly, a generalized Jordan centralizer extends the idea of a centralizer by linking it with a Jordan centralizer, opening pathways to the investigation of new structural properties [11]. These generalizations, while broadening the theoretical landscape, also raise natural questions: under what conditions do these generalized notions collapse back to their classical counterparts? Are these generalizations genuinely more powerful, or do they reveal the inherent rigidity of certain ring structures?

This paper addresses these questions in the setting of torsion-free semiprime rings, which constitute an important class of associative rings widely studied for their structural robustness. Semiprime rings generalize prime rings by allowing a broader algebraic framework while retaining the essential feature that eliminates nilpotent ideals [12]. Torsion-free assumptions, on the other hand, ensure that divisibility constraints do not interfere with the structure, thereby providing a cleaner environment for algebraic characterization [13].

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The main results of this work establish that:

1. Every nonzero generalized Jordan derivation on a torsion-free semiprime ring is, in fact, a derivation mapping the ring into itself.
2. Every generalized Jordan centralizer on a torsion-free semiprime ring coincides with a two-sided centralizer.

These results demonstrate that in the presence of torsion-free semiprimeness, the generalized concepts introduce no genuinely new operator behavior; instead, they reduce to their classical forms. Thus, the work not only strengthens known results in the literature but also highlights the inherent rigidity of semiprime rings under torsion-free restrictions.

The significance of these findings is twofold. First, they contribute to the simplification and classification of operator mappings in semiprime rings, showing that the landscape of possible behaviors is narrower than expected. Second, they provide a solid foundation for future investigations into more general classes of rings, functional identities, and potential applications in operator algebra and mathematical physics.

Main Results

Lemma 1: Lett R be a 2-torsion free semiprime ring, $m,n \geq 0$ be distinct integers with $m + n \neq 0$ and $F: R \rightarrow R$ be a nonzero generalized (m,n) −Jordan derivation with an associated (m,n) −Jordan derivation d. Then, $(\mathrm{m} + \mathrm{n})^2 F(xyx) = m(n-m)F(x)xy + m(3m+n)F(x)yx + m(m-n)F(y)x^2 + 4mnxd(y)x + n(n-m)x^2d(y) + n(m+3n)xyd(x) + n(m-n)yxd(x)$ for all $x,y \in R$.

Lemma 2: Let R be a semiprime ring, $m,n \geq 0$ be distinct integers with $m+n \neq 0$, and $T: R \rightarrow R$ be a nonzero generalized (m,n) −Jordan centralizer with an associated (m,n) −Jordan centralizer C'. Then, $2(m+n)^2T(xyx) = mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 + 2mnxC'(y)x - mnx^2C'(y) + n(m+2n)xyC'(x) + mnyxC'(x)$ for all $x,y \in R$.

Lemma 3: Let R be a semiprime ring and $T: R \rightarrow R$ be an additive mapping. If either $T(x)x = 0 \; or \; xT(x) = 0$ holds for all $x \in R$, then $T = 0$.

Lemma 4: Let R be $mn(m+n)|m-n| - t$orsion free semiprime ring, $m,n \geq 1$ be distinct integers and $d: R \rightarrow R$ be (m,n) −Jordan derivation. Then d is a derivation which maps R into $Z(R)$.

Lemma 5: Let R be an $mn(m+n) -$ torsion free semiprime ring, $m,n \geq 1$ be distinct integers and T: $R \rightarrow R$ be an (m,n) - Jordan centralizer. Then T is a two-sided centralizer.

Theorem 1:

Let R be a k-torsion free semiprime ring, $m,n \geq 1$ be distinct integers, where $k = 6mn(m+n)|m-n|$ and $F: R \rightarrow R$ be a nonzero generalized (m,n)- Jordan derivation. Then F is a derivation which maps R into Z(R).

Proof

Let d be the associated (m,n)- Jordan derivation of F. Since R is a semiprime ring, d is a derivation which maps R into $Z(R)$ (by Lemma 4). Let use the relation $D = F - d$ then, we have

$$(m+n)D(x^2) = (m+n)F(x^2) - (m+n)d(x^2)$$

$$= 2mF(x)x + 2nxd(x) - 2md(x)x - 2nxd(x)$$

$$= 2mD(x)x \text{ for all } x \in R.$$

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Thus,

$$(m + n)D(x^2) = 2mD(x)x \quad (1)$$

Now by replacing x with x^2 in Eq. (1), we get

$$(m + n)D(x^4) = 2mD(x^2)x^2. \quad (2)$$

Multiplying Eq. (2) by $m + n$ we obtain,

$(m + n)^2D(x^4) = 2mD(x^2)x^2(m + n).$

Using Eq. (1) we get,

$$(m + n)^2D(x^4) = 4m^2D(x)x^3, \quad x \in R \quad (3)$$

On the other hand, replacing x^2 with y in Eq. (3) we have,

$$(m + n)^2\mathrm{D}(y^2) = 4m^2D(x)yx, \quad x, y \in R. \quad (4)$$

By using Lemma 1 and the fact that D is a generalized (m, n)- Jordan derivation associated with the zero map as an (m, n)-Jordan derivation we get,

$$(m + n)^2D(x^4) = m(n - m)D(x)x^3 + m(3m + n)D(x)x^3 + m(m - n)D(x^2)x^2$$

$$\text{for all} \quad x \in R. \quad (5)$$

Multiplying both sides of Eq. (5) by 2 we get

$$2(m + n)^2D(x^4) = 2m(n - m)D(x)x^3 + 2m(3m + n)D(x)x^3 + 2m(m - n)D(x^2)x^2$$

$$\text{for all} \quad x \in R. \quad (6)$$

Combining Eq. (2) and Eq. (6), we get

$$2(m + n)^2D(x^4) = 2m(n - m)D(x)x^3 + 2m(3m + n)D(x)x^3$$

$$+(m + n)(m - n)D(x^4), \ x \in R \quad (7)$$

which give

$$(m + n)(m + 3n)D(x^4) = 4m(m + n)D(x)x^3, x \in R. \quad (8)$$

Multiplying both sides of Eq. (8) by $m + n$, we get

$$(m + n)^2(m + 3n)D(x^4) = 4m(m + n)^2D(x)x^3, x \in \quad (9)$$

Multiplying Eq. (3) by $m + 3n$, we get

$$(m + n)^2(m + 3n)D(x^4) = 4m^2(m + 3n)D(x)x^3, \quad x \in R. \quad (10)$$

By comparing (9) and (10), we get

$$4mn(m - n)D(x)x^3 = 0, \quad x \in R. \quad (11)$$

Since R is a $2mn|n - m| -$torsion free ring, $D(x)x^3 = 0$ for all $x \in R$. Applying $D(x)x^3 = 0$, we get $(m + n)^2D(x^4) = 0$, for all $x \in R$. By using the torsion free restriction, we have $D(x^4) = 0$ for all $x \in$

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R. Hence, $D(xy) = D(x)y$ for all $x, y \in R$ (by Lemma 4). This yields $(m+n)D(x)x = 2mD(x)x$ for all $x \in R$. Equivalently, $(m-n)D(x)x = 0$. Since R is an $|m-n|-$ torsion free ring, $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 3, $D = 0$. This completes the proof.

Theorem 2:

Let R be an $6mn(m+n)(2n+m)-$torsion free semiprime ring R, $m, n \geq 1$ be two fixed integers, and suppose $T : R \to R$ be a nonzero generalized $(m+n)$ - Jordan centralizer. Then T is a two-sided centralizer.

Proof

Suppose that C' be the associated $(m+n)$ - Jordan centralizer of T. Since R is a semiprime ring, C' is a two-sided centralizer (by Lemma 5). Let use the relation $D = T - C'$. Then, we have

$$(m+n)D(x^2) = (m+n)T(x^2) - (m+n)C'(x^2)$$

$$= mT(x)x + nxC'(x) - mC'(x)x - nxC'(x)$$

$$= mD(x)x \text{ for all } x \in R. \quad (12)$$

$$(m+n)D(x^2) = mD(x)x, \quad x \in R. \quad (13)$$

Replacing x with x^2 in (13), we get

$$(m+n)D(x^4) = mD(x^2)x^2, \quad x \in R. \quad (14)$$

Multiplying Eq. (14) by $m + n$ and then using Eq. (13), we get

$$(\mathrm{m} + \mathrm{n})^2 D(x^4) = m^2 D(x)x^3 \; x \in R. \quad (15)$$

On the other hand, if we put $y = x^2$ in the relation of Lemma 2, we get

$$2(m+n)^2 D(x^4) = mnD(x)x^3 + m(2m+n)D(x)x^3 - mnD(x^2)x^2, x \in R. \quad (16)$$

Multiplying both sides of Eq. (15) by 2 we get

$$2(m+n)^2 D(x^4) = 2m^2 D(x)x^3, x \in R. \quad (17)$$

Combining Eq. (15) and Eq. (16), we get

$$2(m+n)^2 D(x^4) = mnD(x)x^3 + m(2m+n)D(x)x^3 - n(m+n)D(x^3), x \in R, \quad (18)$$

which Implies

$$(m+n)(2m+3n)D(x^4) = 2m(m+n)D(x)x^3, x \in R. \quad (19)$$

Multiplying both sides of above relation by $m + n$, we have

$$(m+n)^2(2m+3n)D(x^4) = 2m(m+n)^2 D(x)x^3, \quad x \in R. \quad (20)$$

Multiplying Eq. (15) by $(2m + 3n)$, we get

$$(m+n)^2(2m+3n)D(x^4) = m^2(2m+3n)D(x)x^3, \; x \in R \quad (21)$$

By combining Eq. (20) and Eq. (21), we get

$$mn(2n+m)D(x)x^3 = 0, \quad x \in R. \quad (22)$$

Since R is a $mn(2n + m)$-torsion free ring, $D(x)x^3$, = 0 for all $x \in R$.

Applying $D(x)x^3 = 0$ in equation Eq. (15) and then using $(m + n) -$ torsion freeness of R, we get $D(x^4) = 0$. Moreover, since R is a 2 and a 3-torsion free ring, by Lemma 4, we get $D(xy) = D(x)y$ for all $x, y \in R$. Applying this in Eq. (13), yields $(m + n)D(x)x = mD(x)x$ for all $x \in R$. So $nD(x)x = 0$, which implies that $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 3, $D = 0$. This completes the proof.

DISCUSSION

The results obtained in this paper contribute to a rich line of research on derivations, Jordan derivations, and centralizers in associative ring theory. Since Herstein's pioneering work on derivations and Jordan derivations in prime and semiprime rings, much progress has been made in identifying structural conditions under which these mappings exhibit rigidity. One of the most influential findings is that, in semiprime rings, Jordan derivations often coincide with derivations, thereby collapsing the generalized notion into its classical form. This phenomenon has been extensively studied and confirmed in various contexts (see Bresar and Vukman [1], and others on functional identities and operator mappings).

Similarly, centralizers have received significant attention in the literature due to their role in characterizing commutativity and structural constraints of rings. It is well established that in prime and semiprime rings, Jordan centralizers can often be reduced to two-sided centralizers under suitable conditions. This has motivated the exploration of generalized Jordan centralizers, which potentially extend the class of mappings. However, the present results show that in the torsion-free semiprime setting, such generalizations do not lead to fundamentally new behaviors: the mappings collapse back to their two-sided centralizer counterparts.

The contributions of this paper may be viewed from two complementary perspectives:

1. Structural Rigidity of Semiprime Rings. The main theorems highlight that torsion-free semiprime rings exhibit a form of operator rigidity: generalized mappings cannot deviate from their classical definitions. This provides clarity on the scope of admissible operator actions and narrows the classification landscape of mappings on these rings.
2. Extension of Classical Results. Our findings extend several earlier works on Jordan derivations and centralizers, confirming that the torsion-free restriction suffices to eliminate nontrivial counterexamples. By systematically applying functional identities derived from associated mappings and employing torsion-free assumptions, we provide a unified approach that generalizes and strengthens existing results.

From a broader standpoint, these results are valuable in two ways. First, they provide algebraists with simplification tools, showing that within torsion-free semiprime rings, studying generalized derivations and centralizers is equivalent to studying their classical counterparts. Second, they open avenues for further inquiry into whether similar collapses occur in weaker algebraic structures (such as non-associative rings, Lie algebras, or near-rings), or whether torsion restrictions can be relaxed without losing the equivalence.

Moreover, the results also suggest potential applications in operator algebras and functional analysis, where derivations and centralizers play a critical role in studying automorphisms, symmetries, and commutativity-preserving transformations. The elimination of generalized complexity in torsion-free semiprime rings ensures that the focus can be directed toward identifying more nuanced structural behaviors in broader or less restrictive classes of rings.

In summary, the discussion underscores that the present work not only strengthens but also unifies classical results within a generalized framework, reinforcing the inherent rigidity of torsion-free semiprime rings under derivational and centralizer mappings.

CONCLUSION

In this paper, we investigated the behavior of generalized Jordan derivations and generalized Jordan centralizers on torsion-free semiprime rings. Our main theorems establish that every nonzero generalized Jordan derivation in this setting reduces to a derivation mapping the ring into itself, and that every generalized Jordan centralizer coincides with a two-sided centralizer. These findings reveal a structural rigidity: in torsion-free semiprime rings, the generalized concepts collapse into their classical forms.

The implications of these results are noteworthy. First, they streamline the classification of operator mappings in semiprime rings, confirming that generalized Jordan derivations and centralizers do not introduce genuinely new phenomena under torsion-free conditions. Second, they extend classical results on derivations and centralizers by showing that torsion-free restrictions suffice to eliminate pathological behaviors, thereby providing a more general and unified framework.

At the same time, certain limitations and open questions remain. The equivalence established here relies heavily on the torsion-free and semiprime assumptions. It is natural to ask whether similar results can be obtained under weaker conditions, such as in rings with mild torsion, or in broader algebraic systems such as near-rings, Lie algebras, or non-associative rings. Additionally, while our results highlight structural rigidity, further investigation is required to determine how these mappings behave in operator algebras or in the context of functional identities.

Future research may therefore focus on relaxing the torsion-free restriction, exploring analogous results in non-semi-prime or non-associative settings, and applying these findings to operator theory and mathematical physics. In particular, connections with functional identities and commutativity-preserving mappings may yield further insights.

In conclusion, the results presented in this paper reinforce the inherent robustness of torsion-free semiprime rings and provide a clear framework for understanding generalized Jordan derivations and centralizers. By showing that these generalized notions collapse into their classical counterparts, we offer both clarity and simplification, laying the groundwork for further investigations in the structural theory of rings and operator mappings.

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