On the Diophantine Equation

\[(5^n)^x + (4^m p + 1)^y = z^2\]

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Abstract: In this paper, we proved that the Diophantine equation \( (5^n)^x + (4^m p + 1)^y = z^2 \) has no solution in non-negative integers \( x, y, z \) where \( p \) is an odd prime and \( m, n \) is a natural number.

Keywords: Diophantine equations, exponential equations, integer solution.

I. INTRODUCTION

Diophantine equation is one of the significant problems in elementary number theory and algebraic number theory. The Diophantine equation of the type \( a^x + b^y = z^2 \) has been studied by many authors for many years. In 2012, Sroysang [16] proved that the Diophantine equation \( 3^x + 5^y = z^2 \) has a unique non-negative integer solution where \( x, y \) and \( z \) are non-negative integers. The solution \((x, y, z)\) is \((1, 0, 2)\). In the same year, Sroysang [17] proved that the Diophantine equation \( 31^x + 32^y = z^2 \) has no non-negative integer solution.

In 2017, Asthana, S., and Singh, M. M. [3] studies the Diophantine Equation \( 3^x + 13^y = z^2 \) and proved that there are exactly four non-negative integer solutions for \( x, y \) and \( z \). The solutions are \((1, 0, 2)\), \((1, 1, 4)\), \((3, 2, 14)\) and \((5, 1, 16)\) respectively. In 2018, Kumar et al. [10] studied the non-linear Diophantine equations \( 61^x + 67^y = z^2 \) and \( 67^x + 73^y = z^2 \). They proved that these equations have no non-negative integer solution. Additionally, Kumar et al. [11] studied the non-linear Diophantine equations \( 31^x + 41^y = z^2 \) and \( 61^x + 71^y = z^2 \). They determined that these equations have no non-negative integer solution. In the same year, Burshtein et al. [8] examined the solutions to the Diophantine Equation \( M^x + (M + 6)^y = z^2 \) when \( M = 6N + 5 \) and \( M, N \) are primes. They proved that this equation has no solutions.

In 2020, Aggarwal et al. [1] examined the Diophantine equation \( 223^x + 241^y = z^2 \), where \( x, y, z \) are non-negative integers and determined that this equation has no non-negative integer solution.

Moreover, Aggarwal, S. and Sharma, N.[4] investigated the non-linear Diophantine equation \( 379^x + 397^y = z^2 \). The results showed that the considered non-linear Diophantine equation has no non-negative integer solution.

Apart from the above claim, Aggarwal et al. [5] studied the existence of solution of Diophantine equation \( 181^x + 199^y = z^2 \) and proved that this equation has no solution. Similarly, Bhatnagar, K. et al. [7], studied the exponential Diophantine equation and proved that \( 421^p + 439^q = r^2 \) has no solution. In addition, Mishra, R. et al. [14] studied the Diophantine equation \( 211^a + 229^b = y^2 \) and proved that this equation has no solution. In the same year, Kumar, S. et al. [13] investigated the exponential Diophantine equation \( (2^{2m+1} - 1) + (6^r + 1)^3 = a^2 \) and found that this equation has no solution. Kumar, S., et al. [12] also examined the exponential Diophantine equation \( (7^m+1) + (6r + 1)^n = z^2 \) and proved that it has no solution. Moreover, Goel et al. [9] proved that the exponential Diophantine equation \( M^x + M^y = r^2 \) has no solution in whole number.

In 2021, Moonchaisook, V., [15] proved that the non-linear Diophantine equation \( p^x + (p + 4)^y = z^2 \) has no solution. Similarly, Aggarwal, S. [2](2021) studied solutions to the exponential Diophantine equation \( (2^{2m+1} - 1) + 13^n = z^2 \) where \( m, n \) are whole numbers and proved that this equation has no solution in whole number.

Aggarwal, S. et al. [6] investigated the exponential Diophantine equation \( 19^{2m} + (12y + 1)^n = r^2 \) and found no solution in whole number.

Because of this open problem, the author is therefore interested in studying the Diophantine equation; \( (5^n)^x + (4^m p + 1)^y = z^2 \) has no solution in non-negative integers \( x, y, z \) where \( p \) is an odd prime and \( m, n \) is a natural number.

II. PRELIMINARIES

Lemma 1. For every integer \( n \geq 1 \) and \( M, N \) are natural number. Then \( (4^m p + 1)^n = 4N + 1 \)

Proof: Let \( p(n) \) be the proposition that
\[(4^m p + 1)^n = 4N + 1 \text{ for} \text{integer } n \geq 1. \] (1)

1. \( P(1) \) is true. For \( n = 1 \), then \( 4^m p + 1 = 4^m p + 1 \).
2. Show that (1) holds for \( n = k \), Assume (1) holds for \( n = k \),
that is $(4^mp + 1)^k = 4N + 1$ is true. \hspace{2cm} (2)

We consider $P(k+1)$,

$$(4^mp + 1)^{k+1} = (4^mp + 1)(4^mp + 1)^k$$

$$= (4^mp + 1)(4N + 1)$$

$$= 4(4^mpN + 4^{m-1}p + N) + 1$$

Where $4^mpN + 4^{m-1}p + N$ be natural number.

Hence. By induction $P(n)$ is true for integer $n \geq 1$.

**Lemma 2.** The Diophantine equation

$$(5^n)^x + 1 = z^2$$

has no solution in nonnegative integer $x, z$ where $p$ is an odd prime and, $n$ is a natural number.

**Proof:** Suppose that $(5^n)^x + 1 = z^2$ 

$\Rightarrow (5^n)^x = z^2 - 1 = (z - 1)(z + 1)$.

Thus we can fine two non-negative integers $\alpha$ and $\beta$

Such that $(5^n)^x = z - 1$ and $(5^n)^{\beta} = z + 1$ with $\alpha < \beta$ and $\alpha + \beta = x$

Now $(5^n)^{\alpha}((5^n)^{\beta - \alpha} - 1) = 2$

This implies $\alpha = 0$ and $(5^n)^{\beta - \alpha} - 1 = 2$

$\Rightarrow (5^n)^{\beta} = 3$ which is impossible.

Hence the Diophantine equation

$$(5^n)^x + 1 = z^2$$

has no solution.

**Lemma 3.** The exponential Diophantine equation $\alpha

\Rightarrow (4^mp + 1)^{x} = z^2$

has no solution in nonnegative integer $y, z$ where $p$ is an odd prime and, $m$ is a natural number.

**Proof:** Suppose that $1 + (4^mp + 1)^{x} = z^2$

$\Rightarrow (4^mp + 1)^{x} = z^2 - 1 = (z - 1)(z + 1)$.

Thus, we can find two non-negative integers $\alpha$ and $\beta$

Such that $(4^mp + 1)^{\alpha} = z - 1$ and

$(4^mp + 1)^{\beta} = z + 1$ with $\alpha < \beta$ and $\alpha + \beta = y$

Now $(4^mp + 1)^{\alpha}((4^mp + 1)^{\beta - \alpha} - 1) = 2$

This implies $\alpha = 0, \beta = 1$ and

$(4^mp + 1)^{\beta - \alpha} - 1 = 2$

$\Rightarrow 4^mp + 1 = 3 (m \geq 1)$

$\Rightarrow 4^mp = 2$ which is impossible.

Hence the Diophantine equation

$1 + (4^mp + 1)^{x} = z^2$ has no solution.

**III. MAIN THEOREM**

**Theorem 1.** The Diophantine equation

$$(5^n)^x + (4^mp + 1)^{y} = z^2$$

has no solution in non-negative integer $x, y, z$ where $p$ is an odd prime and, $m, n$ are natural numbers.

**Proof:** Suppose that $(5^n)^x + (4^mp + 1)^{y} = z^2$

when $x, y$ and $z$ are non-negative integers, $m$ and $n$ are natural number.

we consider 4 cases including $x = 0$ and $x \geq 1$.

**Case 1.** Suppose that $x = 0, y = 0$.

Thus $z^2 = 2$ which is impossible.

**Case 2.** Suppose that $x = 0, y \geq 1$.

The Diophantine equation $1 + (4^mp + 1)^{y} = z^2$

has no solution in nonnegative integer $y, z$ where $p$ is an odd prime and $m$ is a natural number.

By lemma 3.

**Case 3.** Suppose that $x \geq 1, y = 0$.

The Diophantine equation $(5^n)^x + 1 = z^2$ has no solution in nonnegative integer solution $x, z$ where $p$ is an odd prime and $m, n$ are natural numbers.

By lemma 2.

**Case 4.** Suppose that $x \geq 1, y \geq 1$.

Sine the Diophantine equation

$$(5^n)^x + (4^mp + 1)^{y} = z^2$$

(a) If $x = 2t$ and $y \geq 1$

(b) If $y = 2s$ and $x \geq 1$

(c) If $x = 2t + 1$ and $y = 2s + 1$

(a) If $x = 2t (t > 0$ integer) and $y \geq 1$.

Suppose that $(5^n)^x + (4^mp + 1)^{y} = z^2$

$\Rightarrow (4^mp + 1)^{y} = z^2 - (5^n)^2$.

$\Rightarrow (4^mp + 1)^{y}(z - 5^n)(z + 5^n)$.

Thus we can fine two non-negative integers $\alpha$ and $\beta$

Such that $(4^mp + 1)^{\alpha} = z - 5^n$ and

$(4^mp + 1)^{\beta} = z + 5^n$ with $\alpha < \beta$ and $\alpha + \beta = y$

$\Rightarrow (4^mp + 1)^{\beta} = (4^mp + 1)^{\alpha} + 2(5^n)$.

This implies $(4^mp + 1)[2(5^n)]$ which is impossible.

Hence the Diophantine equation

$(5^n)^x + (4^mp + 3)^{y} = z^2$ has no solution in nonnegative integer solution $x, y, z$ where $p$ is an odd prime and $n$ is a natural number.

(b) If $y = 2s (s > 0$ integer) and $x \geq 1$.

Suppose that $(5^n)^x + (4^mp + 1)^{y} = z^2$.

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\[ (5^n)^x = z^2 - (4^mp + 1)^{2z} \\\n\rightarrow \quad (5^n)^x = (z - (4^mp + 1)^x)(z + (4^mp + 1)^x) \]

Thus, we can find two non-negative integers \( \alpha \) and \( \beta \)
Such that \((5^n)^x = z - (4^mp + 1)^x \) and \((5^n)^y = z + (4^mp + 1)^y\) with \( \alpha < \beta \), \( \alpha + \beta = x \)
\[ \rightarrow \quad (5^n)^\beta = (5^n)^\alpha + 2(4^mp + 1)^{2z} \]

This implies \((5^n)^{2(4^mp + 1)}\) which is impossible.
Hence, the Diophantine equation
\[ (5^n)^x + (4^mp + 3)^y = z^2 \] has no solution in non-negative integer solution \( x, y, z \) where \( p \) is an odd prime and \( n \) is a natural number.

\[ \begin{align*}
c) \text{ If } x &= 2t+1 \ (t \geq 0 \text{ integer}) \quad \text{and} \quad y = 2s+1 \ (s \geq 0 \text{ integer}) \\
\text{Suppose that } (5^n)^x + (4^mp + 1)^y &= z^2 \\
\rightarrow \quad (5^n)^x + (4N + 1) &= z^2, \text{ by lemma 1.} \\
\rightarrow \quad (5^n)^x + 4N &= (z+1)(z-1) \\
\text{Thus, we can find two non-negative integers } \alpha \text{ and } \beta \\
\text{Such that } ((5^n)^x + 4N)^\alpha &= z - 1 \quad \text{and} \\
((5^n)^x + 4N)^\beta &= z + 1 \text{ with } \alpha < \beta, \alpha + \beta = 1 \\
\rightarrow \quad ((5^n)^x + 4N)^{\alpha\beta} &= 2 \\
\text{This implies } \alpha = 0 \text{ and } \beta = 1 \\
\text{Thus } (5^n)^x + 4N = 2 \text{ which is impossible.} \\
\text{Hence, the Diophantine equation} \\
(5^n)^x + (4^mp + 3)^y = z^2 \text{ has no solution in nonnegative integer solution } x, y, z \text{ where } p \text{ is an odd prime and } n \text{ is a natural number.}
\end{align*} \]

**Corollary 1.** The Diophantine equation
\[ (5^n)^x + (4^mp + 1)^y = u^{2n} \]
has no solution, in non-negative integer \( x, y, u \) and \( m, n \) are natural number.

**Proof.** Let \( u^n = z \) then \((5^n)^x + (4^mp + 1)^y = z^2\), which has no solution by Theorem 1.

**Corollary 2.** The Diophantine equation
\[ (5^n)^x + (4^mp + 1)^y = u^{2n+2} \] has no solution, in non-negative integer \( x, y, u \) and \( m, n \) are natural number.

**Proof.** Let \( u^{n+1} = z \)

then \((5^n)^x + (4^mp + 1)^y = u^{2n+2} = z^2\), which has no solution by Theorem 1.

IV. CONCLUSION

The main focus of this paper is to study the solvability of the class of Diophantine equation \((5^n)^x + (4^mp + 1)^y = z^2\) which \( p \) is an odd prime.

The case \( (5^4p + 1) = (5,13) \) was not considered in this work, but through a brief investigation it might be misunderstood that \( 5^{2s+1} + 13^{2t} = z^2 \) is an even. Thus \( z^2 \equiv 0 \mod 3 \) has a solution when \( x \) is an odd number and \( y \) is an even number. But if we proved by using theorem 1 as stated earlier, we will find that \( 5^{2s+1} + 13^{2t} = z^2 \) has no solution.

However, there are still some further points to be considered. There might be other solutions in solving positive integers that need to be investigated.

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REFERENCES


