Block Methods for Solving Second Order Ordinary Differential Equation Using Shifted Linear Multistep Formulas

Utalor I.K*, Durojaye M.O, Ajie Jame I.

Department of Mathematics science, University of Abuja. FCT

***Corresponding Author**

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ABSTRACT:

In this paper, a new block numerical integrator with characteristics of linear multi-step formulas (LMFs) is derived, analyzed, and numerically applied to solve second-order ordinary differential equations (ODEs). It was done by applying a shift operator to two linear multi-step formulae with low Local Truncation Error (LTE). The mathematical derivation of the Linear Multi-step Formulas (LMF) used in the method is based on the method of undetermined coefficients. The analysis of the basic properties of the method shows that the scheme is consistent, convergent, and zero stable. The Numerical experimentation and comparative analysis with existing methods show that our scheme is efficient.

Keywords: One-block Numerical Integrator Method; Shift Operator, Undetermined Coefficients, General Second Order, Initial Value Problems, Self-starting.

INTRODUCTION

Linear multistep methods constitute a powerful class of numerical procedures for solving a second order equation of the form

$$y''(t) = f(t, y(t), y'(t)), y(t_0) = y_0, y'(t_0) = y_1, t \in [a, b] (1)$$

Many Engineers and Scientists spend much of their time working with the models of the world around them. Usually, these mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields such as economics, medicine, and even in areas of biology, just to mention a few. Most often, many of these problems are difficult to be solved by analytical approach. As a result of this, approximate numerical integration methods are routinely used. Conventionally, higher order ODEs are solved by reducing it to a system of first order equations (Lambert, 1973; Fatunla,1988; Brugnano and Trigiante, 1998). The approach results to more cumbersome computation which leads to more errors during the integration process (Adesanya et al. 2009; Vigo-Aguilar et al. 2006).Some attempts have been made to solve problem (1) directly without reduction to a first order systems of equations (Adesanya etal., 2008; Badmus & Yahaya, 2009). Authors such as Okunnuga (2008),Onumanyi et al. (1999), and Omar & Kuboye (2015) to mention a few, have presented continuous linear multistep techniques. To obtain beginning values for their approaches, these authors used the predictor-corrector, block methodology and Taylor series expansion. According to Adesanya, (2011), the predictor corrector method is expensive because subroutines are difficult to construct due to the specific methods required to give starting values and change the step size, resulting in longer computer time and more human labor. The predictors and corrector are not in the same order and as a result of this, the method accuracy is poor. A number of authors including Badmus,et al.(2009), Omar,et al.(2015), Onumanyi et al. (1999) and some others

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have developed the block technique. The undetermined coefficients method was used to construct the coefficients of Boundary Value Methods (BVM) , Linear Multi-step Formula (LMF) with aid of shift operator that possess good stability properties suitable for efficient solutions of stiff initial value problems. These methods have several advantages among which are, the fact that, they do not require any starting value to kick-start their implementation and they are self-starting. It was implemented on first order ordinary differential equations, *(Ajie et al.2019; Ajie et al., 2020)*. This paper is an extension of the work by constructing a method from the perspective of undetermined coefficients method using two Linear Multi-step Formula (LMF) of low Local Truncation Error (LTE) out of the main method and its derivative with the aid of shift operator. We aimed to obtain a block method with the lowest LTE to formulate it and to apply it to solve second order ordinary differential equations directly without reducing it to first order ordinary differential equations.

The present work is outlined as follows: In Section 2, we present the method for solving second order ODEs. The characteristics of the developed formulas are analyzed in Section 3. In Section 4 and 5, we present the numerical results of some test problems to show the efficiency and reliability of the proposed technique. Conclusion is outlined in Section 6.

DERIVATION OF THE SCHEMES

Consider an approximate solution to the general second order ODEs initial value problems by a self-starting block method. The continuous coefficients

$(\{\alpha_j(t)\}_{j=0}^{k}, \{\beta_i(t)\}_{i=0}^{k} and \{\gamma_i(t)\}_{i=0}^{k})$ for the composing LMF are determined by imposing order condition on linear multi-step formula (LMF) and using the method of undetermined coefficients.

Proposition

On the self-starting block methods, let $l, m(\geq 2)$ be integers, also let m denotes the number of k-step LMF in a composite scheme having order of at least $P_{k-j}(\geq 1); j, k(\geq 1)$ .Then, the technique of deriving block methods for solving a second order problem is given by applying shift-operator $l$times, where $l$is given as $l = \frac{2k-m}{m-2}$ *(Utalor et al., 2025).*

PROOF:

The E-operator is effectively applied$l$times on the system of LMF. Thus, there are$2(k+l)$ unknown solution points captured in the block of solution $h^{\lambda} Y_m^{(n)} = (y_{n+1}, y_{n+2}, y_{n+3}, \ldots, hy'_{n+1}, hy'_{n+2}, hy'_{n+3}, \ldots, h^2 y''_{n+1}, \ldots, h^n y^n{}_{n+m})^T$. By this the block definition, $A_1 h^{\lambda} Y_m^{(n)} = h^{\lambda} \sum_{i=0}^{k} A_0 Y_{m-i} + h^{\mu}\left(\sum_{i=1}^{k} B_1 F_m + B_0 F_{m-i}\right); det(A_1) \neq 0;$

is realized if the coefficient A_1, A_0, B_1, B_0, are square matrices of dimensions $2(k+l) \times 2(k+l)$ for a fixed m and $Y_m, Y_{m-i}, F_m and F_{m-i}; m = 0,1,2, \ldots$ are vectors as specified below:

$$h^{\lambda} Y_m^{(n)} = (y_{n+1}, y_{n+2}, y_{n+3}, \ldots, hy'_{n+1}, hy'_{n+2}, hy'_{n+3}, \ldots, h^2 y'^{\,nn}_{n+1}{}_{n+m}{}^{T}$$

$$h^{\lambda} Y_{m-i} = (y_{n+k-1}, y_{n+k-2}, y_{n+k-3}, \ldots, y_n, hy'_{n+k-1}, hy'_{n+k-2}, hy'_{n+k-3}, \ldots, y'_n, h^2 y'^{\,nn}_{n+k-1}{}_{n}{}^{T}$$

$$F_{m-i} = (f_{n+k-1}, f_{n+k-2}, f_{n+k-3}, \ldots, f_n, \ldots, G_{n+k-1}, G_{n+k-2}, G_{n+k-3}, \ldots, G_n, \ldots)^T$$

$$F_m = (f_{n+1}, f_{n+2}, f_{n+3}, \ldots, f_{n+k}, G_{n+1}, G_{n+2}, G_{n+3}, \ldots, G_{n+k})(2)$$

This simply implies that $m + ml = 2(k+l)$ when k is chosen such that l is an integer given as:

$l = \frac{2k-m}{m-2}$; $m \geq 4, k \geq 2$ and $l \geq 0(3)$.

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Where

$$A_1=\begin{pmatrix}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \cdots & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} & \alpha_1'^{(1)} & \alpha_2'^{(1)} & \alpha_3'^{(1)} & \alpha_4'^{(1)} & \cdots & \alpha_k'^{(1)} \\
\alpha_1'^{(1)} & \alpha_2'^{(1)} & \alpha_3'^{(1)} & \cdots & \alpha_{k-1}'^{(1)} & \alpha_k'^{(1)} & \alpha_1''^{(1)} & \alpha_2''^{(1)} & \alpha_3''^{(1)} & \alpha_{41}''^{(1)} & \cdots & \alpha_k''^{(1)} \\
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \cdots & \alpha_{k-1}^{(2)} & \alpha_k^{(2)} & \alpha_1'^{(2)} & \alpha_2'^{(2)} & \alpha_3'^{(2)} & \alpha_4'^{(2)} & \cdots & \alpha_k'^{(2)} \\
\alpha_1'^{(2)} & \alpha_2'^{(2)} & \alpha_3'^{(2)} & \cdots & \alpha_{k-1}'^{(2)} & \alpha_k'^{(2)} & \alpha_1''^{(2)} & \alpha_2''^{(2)} & \alpha_3''^{(2)} & \alpha_4''^{(2)} & \cdots & \alpha_k''^{(2)} \\
\alpha_0^{(1)} & \alpha_1^{(1)} & \alpha_2^{(1)} & \cdots & \alpha_{k-2}^{(1)} & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} & \alpha_0'^{(1)} & \alpha_1'^{(1)} & \alpha_2'^{(1)} & \cdots & \alpha_k'^{(1)} \\
\alpha_0'^{(1)} & \alpha_1'^{(1)} & \alpha_2'^{(1)} & \cdots & \alpha_{k-2}'^{(1)} & \alpha_{k-1}'^{(1)} & \alpha_k'^{(1)} & \alpha_0''^{(1)} & \alpha_1''^{(1)} & \alpha_2''^{(1)} & \cdots & \alpha_k''^{(1)} \\
\alpha_0^{(2)} & \alpha_1^{(2)} & \alpha_2^{(2)} & \cdots & \alpha_{k-2}^{(2)} & \alpha_{k-1}^{(2)} & \alpha_k^{(2)} & \alpha_0'^{(2)} & \alpha_1'^{(2)} & \alpha_2'^{(2)} & \cdots & \alpha_k'^{(2)} \\
\alpha_0'^{(2)} & \alpha_1'^{(2)} & \alpha_2'^{(2)} & \cdots & \alpha_{k-2}'^{(2)} & \alpha_{k-1}'^{(2)} & \alpha_k'^{(2)} & \alpha_0''^{(2)} & \alpha_1''^{(2)} & \alpha_2''^{(2)} & \cdots & \alpha_k''^{(2)} \\
0 & \alpha_0^{(1)} & \alpha_1^{(1)} & \cdots & \alpha_{k-3}^{(1)} & \alpha_{k-2}^{(1)} & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} & \alpha_0'^{(1)} & \alpha_1'^{(1)} & \cdots & \alpha_k'^{(1)} \\
0 & \alpha_0'^{(1)} & \alpha_1'^{(1)} & \cdots & \alpha_{k-3}'^{(1)} & \alpha_{k-2}'^{(1)} & \alpha_{k-1}'^{(1)} & \alpha_k'^{(1)} & \alpha_0''^{(1)} & \alpha_1''^{(1)} & \cdots & \alpha_k''^{(1)} \\
0 & \alpha_0^{(2)} & \alpha_1^{(2)} & \cdots & \alpha_{k-3}^{(2)} & \alpha_{k-2}^{(1)} & \alpha_{k-1}^{(1)} & \alpha_k^{(1)} & \alpha_0'^{(2)} & \alpha_1'^{(2)} & \cdots & \alpha_k'^{(2)} \\
0 & \alpha_0'^{(2)} & \alpha_1'^{(2)} & \cdots & \alpha_{k-3}'^{(2)} & \alpha_{k-2}'^{(2)} & \alpha_{k-1}'^{(2)} & \alpha_k'^{(2)} & \alpha_0''^{(2)} & \alpha_1''^{(2)} & \cdots & \alpha_k''^{(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_0^{(1)} & \alpha_1^{(1)} & \cdot & \cdot & \cdot & \cdot & \cdots & \alpha_k^{(1)} \\
0 & 0 & 0 & \cdots & \alpha_0'^{(1)} & \alpha_1'^{(1)} & \cdot & \cdot & \cdot & \cdot & \cdots & \alpha_k'^{(1)} \\
0 & 0 & 0 & \cdots & \alpha_0^{(2)} & \alpha_1^{(2)} & \cdot & \cdot & \cdot & \cdot & \cdots & \alpha_k^{(2)} \\
0 & 0 & 0 & \cdots & \alpha_0'^{(2)} & \alpha_1'^{(2)} & \cdot & \cdot & \cdot & \cdot & \cdots & \alpha_{.k}'^{(2)}
\end{pmatrix}_{(4k-4)\times(4k-4)}$$

$$B_1=\begin{pmatrix}
\beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} & \cdots & \beta_{k-1}^{(1)} & \beta_k^{(1)} & \gamma_1^{(1)} & \gamma_2^{(1)} & \gamma_3^{(1)} & \gamma_4^{(1)} & \cdots & \gamma_k^{(1)} \\
\beta_1'^{(1)} & \beta_2'^{(1)} & \beta_3'^{(1)} & \cdots & \beta_{k-1}'^{(1)} & \beta_k'^{(1)} & \gamma_1'^{(1)} & \gamma_2'^{(1)} & \gamma_3'^{(1)} & \gamma_4'^{(1)} & \cdots & \gamma_k'^{(1)} \\
\beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} & \cdots & \beta_{k-1}^{(2)} & \beta_k^{(2)} & \gamma_1^{(2)} & \gamma_2^{(2)} & \gamma_3^{(2)} & \gamma_4^{(2)} & \cdots & \gamma_k^{(2)} \\
\beta_1'^{(2)} & \beta_2'^{(2)} & \beta_3'^{(2)} & \cdots & \beta_{k-1}'^{(2)} & \beta_k'^{(2)} & \gamma_1'^{(2)} & \gamma_2'^{(2)} & \gamma_3'^{(2)} & \gamma_4'^{(2)} & \cdots & \gamma_k'^{(2)} \\
\beta_0^{(1)} & \beta_1^{(1)} & \beta_2^{(1)} & & \beta_{k-2}^{(1)} & \beta_{k-1}^{(1)} & \beta_k^{(1)} & \gamma_0^{(1)} & \gamma_1^{(1)} & \gamma_2^{(1)} & & \gamma_k^{(1)} \\
\beta_0'^{(1)} & \beta_1'^{(1)} & \beta_2'^{(1)} & \cdots & \beta_{k-2}'^{(1)} & \beta_{k-1}'^{(1)} & \beta_k'^{(1)} & \gamma_0'^{(1)} & \gamma_1'^{(1)} & \gamma_2'^{(1)} & \cdots & \gamma_k'^{(1)} \\
\beta_0^{(2)} & \beta_1^{(2)} & \beta_2^{(2)} & & \beta_{k-2}^{(2)} & \beta_{k-1}^{(2)} & \beta_k^{(2)} & \gamma_0^{(2)} & \gamma_1^{(2)} & \gamma_2^{(2)} & & \gamma_k^{(2)} \\
\beta_0'^{(2)} & \beta_1'^{(2)} & \beta_2'^{(2)} & \cdots & \beta_{k-2}'^{(2)} & \beta_{k-1}'^{(2)} & \beta_k'^{(2)} & \gamma_0'^{(2)} & \gamma_1'^{(2)} & \gamma_2'^{(2)} & \cdots & \gamma_k'^{(2)} \\
0 & \beta_0^{(1)} & \beta_1^{(1)} & \cdots & \beta_{k-3}^{(1)} & \beta_{k-2}^{(1)} & \beta_{k-1}^{(1)} & \beta_k^{(1)} & \gamma_0^{(1)} & \gamma_1^{(1)} & \cdots & \gamma_k^{(1)} \\
0 & \beta_0'^{(1)} & \beta_1'^{(1)} & \cdots & \beta_{k-3}'^{(1)} & \beta_{k-2}'^{(1)} & \beta_{k-1}'^{(1)} & \beta_k'^{(1)} & \gamma_0'^{(1)} & \gamma_1'^{(1)} & \cdots & \gamma_k'^{(1)} \\
0 & \beta_0^{(2)} & \beta_1^{(1)} & \cdots & \beta_{k-3}^{(2)} & \beta_{k-2}^{(2)} & \beta_{k-1}^{(2)} & \beta_k^{(2)} & \gamma_0^{(2)} & \gamma_1^{(2)} & \cdots & \gamma_k^{(2)} \\
0 & \beta_0''^{(2)} & \beta_1'^{(2)} & \cdots & \beta_{k-3}'^{(2)} & \beta_{k-2}'^{(2)} & \beta_{k-1}'^{(2)} & \beta_k'^{(2)} & \gamma_0'^{(2)} & \gamma_1'^{(2)} & \cdots & \gamma_k'^{(2)} \\
\vdots & \vdots & & & & & & & & & & \vdots \\
0 & 0 & & & \beta_0^{(1)} & \beta_1^{(1)} & & & & & & \beta_k^{(1)} \\
0 & 0 & \cdot & \cdot\ \cdot\ 0 & \beta_0'^{(1)} & \beta_1'^{(1)} & & & & & & \beta_k'^{(1)} \\
0 & 0 & & & \beta_0^{(2)} & \beta_1^{(2)} & & & & & & \beta_k^{(2)} \\
0 & 0 & \cdot & \cdot\ \cdot\ 0 & \beta_0'^{(2)} & \beta_1'^{(2)} & & & & \cdot & \cdot\ \cdot & \beta_k'^{(2)}
\end{pmatrix}_{(4k-4)\times(4k}$$

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$$A_0 = \begin{pmatrix} 0 & 0 & . & \alpha_0^{(1)} & . & 00 & \alpha'^{(1)}_0 \\ 0 & 0 & . & .\alpha'^{(1)}_0 & . & 00 & \alpha'^{(1)}_0 \\ 0 & 0 & . & .\alpha_0^{(2)} & . & 00 & \alpha'^{(2)}_0 \\ . & . & & \alpha'^{(2)}_0 & & . & .\alpha'^{(2)}_0 \\ . & . & & & & . & . \\ . & . & & & & . & . \\ 0 & 0 & . & . & . & 00 & 0 \end{pmatrix}_{(4k-4)\times(4k-4)}, B_0 = \begin{pmatrix} 0 & 0 & . & \beta_0^{(1)} & . & 00 & \gamma_0^{(1)} \\ 0 & 0 & . & .\beta'^{(1)}_0 & . & 00 & \gamma'^{(1)}_0 \\ 0 & 0 & . & \beta\alpha_0^{(2)} & . & 00 & \gamma_0^{(2)} \\ . & . & & \beta'^{(2)}_0 & & . & .\gamma'^{(2)}_0 \\ . & . & & & & . & . \\ . & . & & & & . & . \\ 0 & 0 & . & . & . & 00 & 0 \end{pmatrix}_{(4k-4)\times(4k-4)}$$

(4)

In this case, the number of equations is m=4 . *(Utalor et al., 2025).*

Main Formulas (k=3)

Let us consider the Linear Multi-step formula (LMF) of the form

$y_{n+j} = y_{n+i} + y'_{n+i} + h^2 \sum_{j=0}^{k} \beta_j^{(1)} f_{n+j} + h^3 \sum_{j=0}^{k} \gamma_j^{(1)} G_{n+j}, j = 0,1,\ldots,k, i = 0$ (5) using undetermined coefficients method. This leads to the following matrix equation:

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ i & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \frac{i^2}{2!} & i & 1 & \ldots & 1 & 0 & \ldots & 0 \\ \frac{i^3}{3!} & \frac{i^2}{2!} & 0 & \ldots & k & 1 & \ldots & 1 \\ . & . & . & \ldots & . & . & \ldots & . \\ . & . & . & \ldots & . & . & \ldots & . \\ . & . & . & \ldots & . & . & \ldots & . \\ \frac{i^q}{q!} & \frac{i^{q-1}}{q-1!} & 0 & \ldots & \frac{k^{q-2}}{q-2!} & 0 & \ldots & \frac{k^{q-3}}{q-3!} \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha'_i \\ \beta_0 \\ . \\ . \\ . \\ \beta_k \\ \gamma_0 \\ . \\ . \\ . \\ \gamma_k \end{pmatrix} = \begin{pmatrix} 1 \\ j \\ \frac{j^2}{2!} \\ \frac{j^3}{3!} \\ . \\ . \\ . \\ \frac{j^q}{q!} \end{pmatrix} \quad (6)$$

When $j = t$, equation (6) is solved using the Mathematica software package to obtain the value of the continuous coefficient $\alpha_i(t), \beta_i(t), G_i(t)$, expressed as functions of t and can be written as:

$y_{n+j} = \alpha_i y_{n+i} + \alpha'_i h y'_{n+i} + h^2(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3}) + h^3(\gamma_0 G_n + \gamma_1 G_{n+1} + \gamma_2 G_{n+2} + \gamma_3 G_{n+3})$ where:

$$\alpha_i(t) = 1$$
$$\alpha'_i(t) = t$$
$$\beta_0(t) = \frac{t^2}{2} - \frac{1397t^4}{2448} + \frac{21187t^5}{36720} - \frac{281t^6}{1080} + \frac{625t^7}{11016} - \frac{41t^8}{8568} - \frac{t^9}{66096}$$

$$\beta_1(t) = \frac{189\,t^4}{748} - \frac{207\,t^5}{3740} - \frac{31\,t^6}{440} + \frac{265\,t^7}{7854} - \frac{169\,t^8}{41888} - \frac{t^9}{26928}$$
$$\beta_2(t) = \frac{924\,t^4}{2992} - \frac{7587\,t^5}{14960} + \frac{141\,t^6}{440} - \frac{2743\,t^7}{31416} + \frac{89t^8}{10472} + \frac{t^9}{26928}$$
$$\beta_3(t) = \frac{5\,t^4}{612} - \frac{133\,t^5}{9180} + \frac{11\,t^6}{1080} - \frac{61\,t^7}{19278} + \frac{11\,t^8}{34272} + \frac{t^9}{66096}$$
$$\gamma_0(t) = \frac{t^3}{6} - \frac{467\,t^4}{1683} + \frac{28531\,t^5}{134640} - \frac{113\,t^6}{1320} + \frac{4987\,t^7}{282744} - \frac{181\,t^8}{125662} - \frac{t^9}{242352}$$
$$\gamma_1(t) = -\frac{93\,t^4}{187} + \frac{1299\,t^5}{1870} - \frac{25\,t^6}{66} + \frac{1465\,t^7}{15708} - \frac{89\,t^8}{10472} - \frac{t^9}{26928}$$
$$\gamma_2(t) = -\frac{318\,t^4}{1496} + \frac{3099\,t^5}{14960} - \frac{181\,t^6}{1320} + \frac{1247\,t^7}{31416} - \frac{169\,t^8}{41888} - \frac{t^9}{26928}$$
$$\gamma_3(t) = \frac{t^4}{3366} - \frac{t^5}{1683} + \frac{t^6}{1980} - \frac{31\,t^7}{141372} + \frac{t^8}{20944} - \frac{t^9}{242352} \quad (7)$$

Differentiating (7) in respect to t gives

$y'_{n+j} = \alpha'_i y_{n+i} + \alpha$ $\;'\,'\,'\,'\,'\,'\,'\,2'\,'\,'\,'$ $_{i\,n+i\,0\,n\,1\,n+1\,2\,n+2\,3\,n+3}$ $_{0\,n\,1\,n+1\,2\,n+2\,3\,n+3}$ where:

$$\alpha'_i(t) = 0$$
$$\alpha''_i(t) = 1$$
$$\beta'_0(t) = t - \frac{1397\,t^3}{612} - \frac{21187t^4}{7344} - \frac{281t^5}{180} + \frac{4375t^6}{11016} - \frac{41t^7}{1071} - \frac{t^8}{7344}$$
$$\beta'_1(t) = \frac{189\,t^3}{187} - \frac{204\,t^4}{748} - \frac{93\,t^5}{220} + \frac{265\,t^6}{1122} - \frac{169\,t^7}{5236} - \frac{t^8}{2992}$$
$$\beta'_2(t) = \frac{927\,t^3}{748} - \frac{7587\,t^4}{2992} + \frac{423\,t^5}{220} - \frac{2743\,t^6}{4488} - \frac{89\,t^7}{1309} + \frac{t^8}{2992}$$
$$\beta'_3(t) = \frac{5\,t^3}{153} - \frac{133\,t^4}{1836} + \frac{11\,t^5}{180} - \frac{61\,t^6}{2754} + \frac{11\,t^7}{4284} + \frac{t^8}{7344}$$
$$\gamma'_0(t) = \frac{t^2}{2} - \frac{1868\,t^3}{1683} + \frac{28531\,t^4}{26928} - \frac{113\,t^5}{220} + \frac{4987\,t^6}{40392} - \frac{181\,t^7}{15708} - \frac{t^8}{26928}$$
$$\gamma'_1(t) = -\frac{372\,t^3}{187} + \frac{1299\,t^4}{374} - \frac{25\,t^5}{11} + \frac{1465\,t^6}{2244} - \frac{89\,t^7}{1309} - \frac{t^8}{2992}$$
$$\gamma'_2(t) = -\frac{183\,t^3}{374} + \frac{3099\,t^4}{2992} - \frac{181\,t^5}{220} + \frac{1247\,t^6}{4488} - \frac{169\,t^7}{5236} - \frac{t^8}{2992}$$
$$\gamma'_3(t) = \frac{2\,t^3}{1683} - \frac{5\,t^4}{1683} + \frac{t^5}{330} - \frac{31\,t^6}{20196} + \frac{t^7}{2618} - \frac{t^8}{26928} \quad (8)$$

Evaluating in (7 and 8) at the points $t = 3,2,1$ gives the method and its derivative.

It is good to note that, traditionally, the continuous method (7) and its derivatives y'(t) are used to produce the main and additional methods which are combined to provide all approximations on the entire interval for value problems as stated in (1). But in the case of shifted method presented below, we used two methods out of the main methods *(t=2 and 1)* and its derivative to form the one block method. The choice is based on the ones that have the lowest LTE. We then have:

$$y_{n+2} = y_n + 2hy'_n + h^2\left(\frac{102719}{144585}f_n + \frac{60644}{58905}f_{n+1} + \frac{15091}{58905}f_{n+2} + \frac{556}{144585}f_{n+3}\right)$$

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$$+h^3(\frac{43814}{530145}G_n - \frac{13408}{58905}G_{n+1} - \frac{4432}{58905}G_{n+2} + \frac{8}{75735}G_{n+3})$$
$$y_{n+1} = y_n + hy_n^{'} + h^2(\frac{689539}{2323360}f_n + \frac{295073}{1884960}f_{n+1} + \frac{41801}{942480}f_{n+2} + \frac{4807}{4626720}f_{n+3})$$
$$+h^3(\frac{15355}{484704}G_n - \frac{91169}{942480}G_{n+1} - \frac{31429}{1884960}G_{n+2} + \frac{137}{4241160}G_{n+3})$$
$$y_{n+2}^{'} = y_n^{'} + h(\frac{20467}{48195}f_n + \frac{20212}{19635}f_{n+1} + \frac{10643}{19635}f_{n+2} + \frac{188}{48195}f_{n+3})$$
$$+h^2(\frac{9553}{176715}G_n - \frac{296}{3927}G_{n+1} - \frac{2041}{19635}G_{n+2} + \frac{16}{176715}G_{n+3})$$
$$y_{n+1}^{'} = y_n^{'} + h(\frac{77089}{192780}f_n + \frac{161731}{314160}f_{n+1} + \frac{6551}{78540}f_{n+2} + \frac{1469}{771120}f_{n+3})$$
$$+h^2(\frac{33841}{706860}G_n - \frac{12833}{62832}G_{n+1} - \frac{2437}{78540}G_{n+2} + \frac{163}{2827440}G_{n+3})(9)$$

Here, the number of unknowns $y_{n+j}, y'_{n+j}, j = 1,2,3$is greater than the number of equations in (9). Originally, there are four equations in equation (9). By applying the theory given in equation (3) on equation (9), where $(k = 3, m = 4)$, the result obtained requires the shift operator once. Shifting once, additional four equations are added which makes it eight equations now with eight unknowns ($y_{n+j}, y_{n+j}^{'}, j = 1,2,3,4$). Since the number of unknowns and the number of equations are equal, then the coefficients of the resultant block method (4) after the shift operator application in vector form are given below:

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{102719}{144585} & 0 & 0 & 0 & \frac{43814}{530145} \\ 0 & 0 & 0 & \frac{689539}{2313360} & 0 & 0 & 0 & \frac{15355}{484704} \\ 0 & 0 & 0 & \frac{20467}{48195} & 0 & 0 & 0 & \frac{9553}{176715} \\ 0 & 0 & 0 & \frac{77089}{192780} & 0 & 0 & 0 & \frac{33841}{706860} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \frac{60644}{58905} & \frac{15091}{58905} & \frac{556}{144585} & 0 & \frac{-13408}{58905} & \frac{-4432}{58905} & \frac{8}{75735} & 0 \\ \frac{295073}{1884960} & \frac{41801}{942480} & \frac{4807}{4626720} & 0 & \frac{-91169}{942480} & \frac{-31429}{1884960} & \frac{137}{4241160} & 0 \\ \frac{20212}{19635} & \frac{10643}{19635} & \frac{188}{48195} & 0 & \frac{-296}{3927} & \frac{-2041}{19635} & \frac{16}{176715} & 0 \\ \frac{161731}{314160} & \frac{6551}{78540} & \frac{1469}{771120} & 0 & \frac{-12833}{62832} & \frac{-2437}{78540} & \frac{163}{2827440} & 0 \\ \frac{102719}{144585} & \frac{60644}{58905} & \frac{15091}{58905} & \frac{556}{144585} & \frac{43814}{530145} & \frac{-13408}{58905} & \frac{-4432}{58905} & \frac{8}{75735} \\ \frac{689539}{2313360} & \frac{295073}{1884960} & \frac{41801}{942480} & \frac{4807}{4626720} & \frac{15355}{484704} & \frac{-91169}{942480} & \frac{-31429}{1884960} & \frac{16}{176715} \\ \frac{20467}{48195} & \frac{20212}{19635} & \frac{10643}{19635} & \frac{188}{48195} & \frac{9553}{176715} & \frac{-296}{3927} & \frac{-2041}{19635} & \frac{-52}{2835} \\ \frac{77089}{192780} & \frac{161731}{314160} & \frac{6551}{78540} & \frac{1469}{771120} & \frac{33841}{706860} & \frac{-12833}{62832} & \frac{-2437}{78540} & \frac{163}{2827440} \end{bmatrix}$$

(10) .

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FUNDAMENTAL PROPERTIES OF THE METHOD

Order and Local truncation error

The linear differential operator $L[z(x); h]$ associated with the block (10) is defined by $L[y(t);h] = A_1 h^{\lambda} Y_m^{(n)} - h^{\lambda} \sum_{i=0}^{k} A_0 Y_{m-i} + h^{\mu}\left(\sum_{i=1}^{k} B_1 F_m + B_0\, F_{m-i}\right)$ (11) Expanding (10) using Taylor series,we obtained

$$L[y(t);h] = C_0 y(t) + C_1 h y'(t) + C_2 h^2 y''(t) + \ldots + C_q h^q y^{(q)}(t) + \ldots (12)$$

$where C_q, q = 0,1,2,\ldots are cons\tan t s given in terms of \alpha_j, \beta_j and \lambda_j$ So that

$$L[y(t);h] = C_{p+2} h^{p+2} y^{(p+2)}(t) + 0(h^{p+3}) with p = 8$$

$where C_0 = C_1 = C_2 = \ldots = C_{P+1} = 0 and C_{10} \neq 0 = C_{p+2}$. In this case, p is the order and C_{p+2} is the error constant (*Lambert, 1973*).The error constants $C_{p+2}^{T_J^{(v)}}$ for $j = 1,2 and v = 0,1$ are given below.

$$C_{p+2}^{T_J^{(v)}} = \left(-\frac{1}{14025}, -\frac{137}{6283200}, -\frac{2}{32725}, -\frac{163}{4188800}\right)^T$$

Consistency

Definition 1: A block method is said to be consistent if its order is greater than one (*Lambert, 1973*). It follows that for all the formulae has order $p = 8$. Since the order of the formulae is greater than one, it means that they are consistent.

Zero Stability

Definition 2:The implicit block method (10) is said to be zero stable if the roots z_s, s=1,…,n of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z) = det\left[z\overline{A_1} - A_0\right]$$

satisfies $/z_s/\leq 1$ and every root with $/z_s/=1$ has multiplicity not exceeding two in the limit as $h\rightarrow 0$ (*Henrici, 1962*). Using the definitions, the method in (10) may be rewritten in a more appropriate vector form to study zero-stability as

$$A_1 y_m^{(n)} - A_0 y_{m-1} = 0 where y_m^{(n)} = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T, y_{m-i} = (y_{n-1}, y_{n-2}, \ldots, y_n)^T$$
$$and A_1, A_0 are cons\tan d matrrice given by$$

$$we have \ddot{\rho}(z) = det\left[z\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right]$$

$$= -2z^3 = 0$$
$$\therefore z = 0$$

Convergence

Theorem: Consistency and zero stability are sufficient conditions for linear multistep method to be convergent *(Henrici, 1962).* Since the method (10) is consistent and zero stable, it implies that the method is convergent .

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NUMERICAL ILLUSTRATIONS

In order to ascertain the efficiency of the developed method, numerical experiment of some problems are performed and the results compared with that of the earlier once in the literature. The accuracy is measured by using the following formulae:$Erc = \|y(t_j) - y_j\|$, where Erc denotes the absolute error at the considered node, $y(t_j)$ and $y_j$are exact and approximate solutions of the problems, respectively.The computed results for the three problems using the methods proposed are presented in tables and graphically.

Problem 1. Application problem in physical sciences and engineering: Mass Spring Motion.

Consider an application where a 128lb weight is attached to a spring having a spring constant of 64lb/ft. The weight is set in motion with no initial velocity by displacing it 6 *inches* above the equilibrium position and by simultaneously applying to the weight an external force, $F(t) = 8\,sin(4t)$. Assuming no air resistance, compute the subsequent motion of the weight at $t: 0.01 \leq t \leq 0.10$.

Modeling this problem into a mathematical equation and applying the proposed method to compute the motion on the weight attached to the spring. The following parameters were considered

$$m = 128, k = 64, b = 0 and F(t) = 0.$$

Thus, the application problem is written mathematically

$$\frac{d^2x}{dt^2} + 16x = 2\,sin(4t), x(0) = -\frac{1}{2}, x'(0) = 0$$

with the analytical solution given below:

$x(t) = -\frac{1}{2}cos(4t) + \frac{1}{16}sin(4t) - \frac{t}{4}cos(4t)$*(Ogunware et al., 2023).*

The new method is applied to solve problem 1 and then compared the result with

Ogunware et al., (2023) and *Skwame et al., (2018)* as shown below:

Table 1. Comparison of Absolute errors for problem 1(h = 0.1)

t	Exact solution	Computed solution	Error	Ogunware et al., (2023),	Skwame et al., (2018),
0.1	-0.499598720210477	-0.499598720223385	1.2908e-11	3.0489e-11	1.6621e-09
0.2	-0.498390193309750	-0.498390193339769	3.0019e-11	6.1223e−11	1.1587e-08
0.3	-0.496368369740280	-0.496368369787363	4.7083e-11	9.2137e-11	2.9743e-08
0.4	-0.493528526608179	-0.493528526672251	6.4072e-11	1.2316e-10	8.6076e-08
0.5	-0.489867287968945	-0.489867288029452	6.0507e-11	1.5423e-10	9.0505e-08
0.6	-0.485382642897099	-0.485382642947256	5.0157e-11	1.8526e-10	1.3291e-07
0.7	-0.480073961290567	-0.480073961330294	3.9727e-11	2.1620e-10	1.8317e-07
0.8	-0.473942007364362	-0.473942007393595	2.9233e-11	2.4698e-10	2.4110e-07
0.9	-0.466988950792028	-0.466988950739920	5.2108e-11	2,7751e-10	3.0654e-07
1	-0.459218375457224	0.459218375300726	1.56498e-10	3.0773e-10	3.7922e-07

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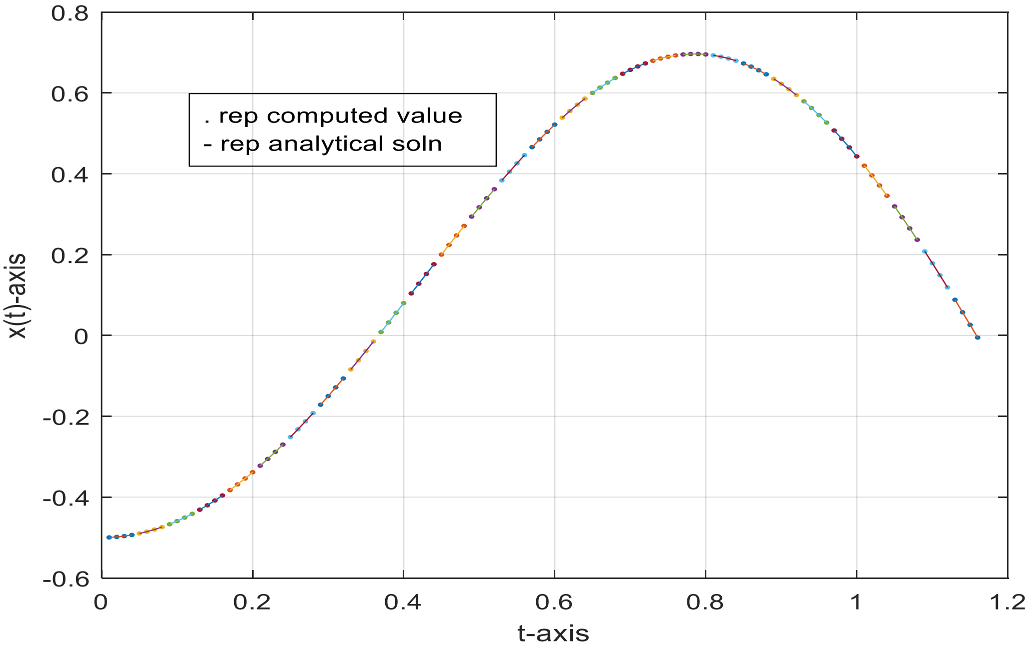

Fig. 1. Plots of analytical and computed solution for Problem (1).

It shows good agreement between the numerical and exact solutions

Problem 2. Consider a highly stiff initial value problem

$$f(t,y,y') = 100y, y(0) = 0, y'(0) = -10$$

Exact solution

$$y(t) = e^{-10t}, withh = \frac{1}{100}$$

as described in*(Opeyemi et al, 2024; Akeremale et al, 2024).*

The new method is applied to solve problem 2 and then compared the result with

(Opeyemi et al, 2024; Akeremale et al, 2024) as shown below:

Table 2. Comparing the absolute errors in the new methods to others(problem 2)

t	**Exact solution**	**Computed solution**	**Error**	**Opeyemi et al., (2024),**	**Akeremale et al., (2024),**
0.01	0.904837418035960	0.904837418035940	2.00000e-14	1.791916e-9	2.04 e-7
0.02	0.818730753077982	0.818730753077918	6.40000e-14	4.313791e-9	1.85e-7
0.03	0.740818220681718	0.740818220681572	1.46000e-13	6.867877e-9	1.68e-7
0.04	0.670320046035639	0.670320040300624	5.735015e-9	8.831792e-9	1.52 e-7
0.05	0.606530659712633	0.60653064316728	1.654535e-8	2.292857e-8	1.38e-7
0.06	0.548811636094027	0.54881162053084	1.556317e-8	3.925958e-8	1.26e-7
0.07	0.496585303791409	0.49658537342543	6.963402e-8	5.779747e-8	1.15 e-7
0.08	0.449328964117222	0.44932891673509	4.738213e-8	7.855516e-8	1.05 e-7

0.09	0.406569659740599	0.40657004499466	8.395048e-7	1.0158421e-7	5.70e-4
0.10	0.367879441171442	0.36788015032909	7.091575e-7	1.2697376e-7	5.17 e-4

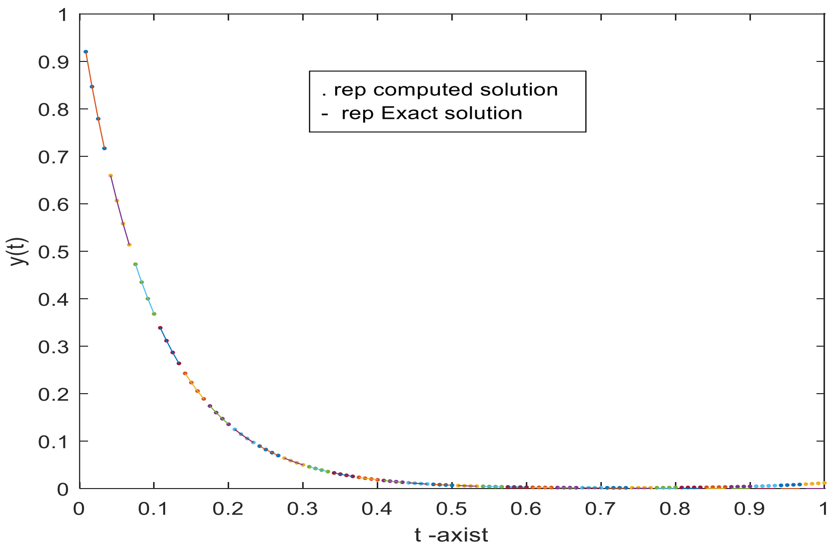

Fig. 2. Plots of exact and computed solution for Problem 2

Problem 3. Consider a highly oscillatory test problem

$$y'' + \lambda^2 y = 0 \lambda = 2 and y(0) = 1, y'(0) = 2$$

Exact solution

$$y(t) = cos\, 2\, t + sin\, 2\, t$$

(Opeyemi et al, 2024).

The new method is applied to solve problem 3 and then compared the result with

Opeyemietal., (2024) and Omole et al., (2018) as shown below:

Table 3: Comparison of absolute error for Problem 3(h = 1/100)

t	Exact solution	Computed solution	Error	Opeyemie tal., (2024),	Omole et al., (2018),
0.01	1.019798673359911	1.019798673359911	0.0000e-0	2.2890e-13	3.409e-11
0.02	1.039189440847612	1.039189440847612	0.00000e-0	6.5075e-13	3.239e-11
0.03	1.058164546414649	1.058164546414649	0.00000e-0	6.2638e-13	3.465e-11
0.04	1.076716400271792	1.076716400129661	1.42131e-10	1.63787e-12	2.40e-13
0.05	1.094837581924854	1.094837586477904	4.55305e-9	3.88273e-12	1.780e-12
0.06	1.112520843142786	1.112520852389196	9.24641e-9	6.57699e-12	7.467e-11
0.07	1.129759110856874	1.129759124792946	1.393607e-8	9.70585e-12	3.904e-11
0.08	1.146545489989873	1.146545508672413	1.868254e-8	1.32541e-12	4.132e-11
0.09	1.162873266213946	1.162873287508854	2.129490e-8	1.72064e-11	1.197e-10
0.10	1.178735908636303	1.178735932535063	2.389876e-8	2.15469e-11	8.342e-11

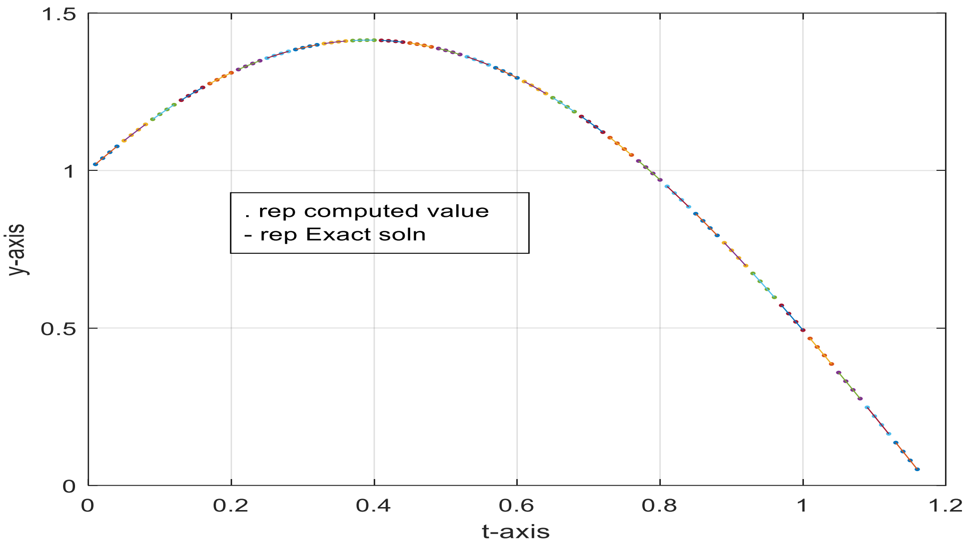

Fig. 3. Plots of exact and computed solution for Problem 3

DISCUSSION OF RESULTS

The results obtained from the three test problems considered are summarized in Tables 1 - 3 and Figs. 1 - 3. In order to examine the usefulness and applicability of the new method, in table 1, we presented the numerical results and comparison of the absolute error with (Ogunware et al., (2023); Skwame et al., 2018) using the same order and values of h. It is very obvious that the new method is advantageous over other existing methods which solved the same problems in the literature. Table 2 shows the comparison of the results of our method with that of Opeyemi et al., (2024) and Akeremale et al., (2024) for problem 2. It is obvious that our method compares favorably with the existing methods despite their different methods. Table 3 shows the results generated for test problem 3 using the new method and that of Opeyemi et al., (2024) and Omole et al., (2018). The new method gives a minimal error. In general, figure 1-3 show that the implementation of the new scheme on three specific examples demonstrates its favorable comparison with exact solutions.

CONCLUSION

This paper has demonstrated how self-starting block methods for the solution of second order can be constructed by applying shift operator on two different linear multi-step formulae with low LTE. The continuous coefficients of the linear multi-step methods are derived using the undetermined coefficients method. Three problems in literature were used to show the efficiency of the methods and accuracy when compared to other methods. It was observed that the Block Methods for solving second order Ordinary Differential Equation using Shifted Linear Multi-step Formulae compare favorably well with some existing methods cited in the literature and Exact solution. The problem was solved using Matlab, not an in-build code but a code written by the authors.

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