Bitopological Harmonious Labeling of Some Star Related Graphs

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ABSTRACT

Bitopological harmonious labeling for a graph $G = (V(G), E(G))$ with n vertices, is an injective function $f: V(G) \to 2^X$, where X is any non – empty set such that $|X| = m,\ m < n$ and $\{f(V(G))\}$ forms a topology on X, that induces an injective function $f^*: E(G) \to 2^{X^*}$, defined by $f^*(uv) = f(u) \cap f(v)$ for every $uv \in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^* = X \backslash \{1,2, \dots, m\}$. A graph that admits bitopological harmonious labeling is called a bitopological harmonious graph. In this paper, we discuss bitopological harmonious labeling of some star related graphs.

Keywords: Bitopological harmonious graph, bistar graph, spider graph, lilly graph, firecracker graph.

INTRODUCTION

In this paper we consider only simple, finite and undirected graphs. The graph G has a vertex set V = V(G) and edge set E = E(G). For notations and terminology we refer to Bondy and Murthy[2]. Acharya [1] established another link between graph theory and point set topology. Selestin Lina S and Asha S defined bitopological star labeling for a graph $G = (V, E)$ as X be any non-empty set if there exists an injective function $f: V(G) \to 2^X$ which induces the function $f^*: E(G) \to 2^X$ as $f^*(v_1 v_2) = [f(v_1) \cup f(v_2)]^c$ for every $v_1, v_2 \in V(G)$, if $\{f(V(G))\}$and $\{f^*(E(G))\} \cup X$ are topolologies on X then G is said to be bitopological star graph. In this paper we proved some star related graphs are bitopological harmonious graph.

Basic Definitions

Definition 2.1

Bitopological harmonious labeling of a graph $G = (V(G), E(G))$ with n vertices is an injective function $f: V(G) \to 2^X$, where X is any non – empty set such that $|X| = m,\ m < n$ and $\{f(V(G))\}$ forms a topology on X, that induces an injective function $f^*: E(G) \to 2^{X^*}$, defined by $f^*(uv) = f(u) \cap f(v)$ for every $uv \in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^* = X \backslash \{1,2, \dots, m\}$. A graph that admits bitopological harmonious labeling is called a bitopological harmonious graph.

Definition 2.2

Bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.

Definition 2.3

A spider graph $SP(1^n 2^{2m})$ is a star graph $K_{1,n+m}$ such that each of which m vertices is joined to new vertex.

Definition 2.4

Lilly graph L_n, $n \geq 2$, is obtained from 2 stars $2K_{1,n}, n \geq 2$, by joining 2 paths $2P_n, n \geq 2$ with sharing a common vertex.

Definition 2.5

Fire cracker graph $F_{n,k}$ is the graph obtained by concatenation of n $k-$ stars by linking one leaf from each.

MAIN RESULTS

Theorem 3.1

The bistar graph $B_{m,n}, m, n \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = B_{m,n}$.

Let $V(G) = \{u, v\} \cup \{u_i/1 \leq i \leq m\} \cup \{v_i/1 \leq i \leq n\}$.

Let $E(G) = \{uv\} \cup \{uu_i/1 \leq i \leq m\} \cup \{vv_i/1 \leq i \leq n\}$.

$|V(G)| = m + n + 2, |E(G)| = m + n + 1$.

Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \to 2^X$ as follows:

$f(u_1) = \phi$;

$f(u_i) = \{1,2, \dots, i - 1\}$ for $2 \leq i \leq m$;

$f(u) = \{1,2, \dots, m\}$;

$f(v_i) = \{1,2, \dots, m + i\}$ for $1 \leq i \leq n$;

$f(v) = \{1,2, \dots, m + n + 1\}$.

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \to X^*$ is given as follows:

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

$f^*(uu_i) = f(u_i)$ for $1 \leq i \leq m$.

$f^*(vv_i) = f(v_i)$ for $1 \leq i \leq n$.

$f^*(uv) = f(v)$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

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Example 3.2

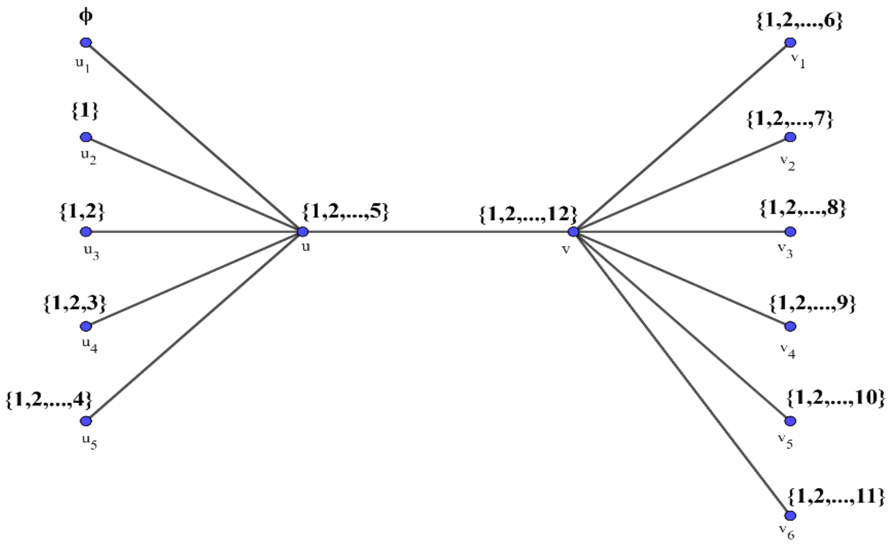

Fig 3.1 Bitopological harmonious labeling of $B_{5,6}$

Theorem 3.3

The Spider graph $SP(1^n 2^m)$, $m, n \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = SP(1^n 2^m)$.

Let $V(G) = \{v_i, u_j / 0 \leq i \leq n, 1 \leq j \leq 2m\}$ where v_0 be the centre vertex.

Let $E(G) = \{v_0 v_i / 1 \leq i \leq n\} \cup \{v_0 u_{2i-1} / 1 \leq i \leq m\}\} \cup \{u_{2i-1} u_{2i} / 1 \leq i \leq m\}$. .

$Then\ |V(G)| = n + 2m + 1, |E(G)| = n + 2m.$

Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ as follows:

$f(v_1) = \phi;$

$f(v_i) = \{1,2, \dots, i-1\}$ for $2 \leq i \leq n;$

$f(u_{2i}) = \{1,2, \dots, n + 2i - 2\}$ for $1 \leq i \leq m;$

$f(u_{2i-1}) = \{1,2, \dots, n + 2i - 1\}$ for $1 \leq i \leq m;$

$f(v_0) = \{1,2, \dots, n + 2m\}.$

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ is given as follows:

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

Here $f^*(v_0 v_i) = f(v_i)$ for $1 \leq i \leq n;$

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$f^*(v_0u_{2i-1}) = f(u_{2i-1})$ for $1 \le i \le m$;

$f^*(v_{2i-1}u_{2i}) = f(u_{2i})$ for $1 \le i \le m$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.4

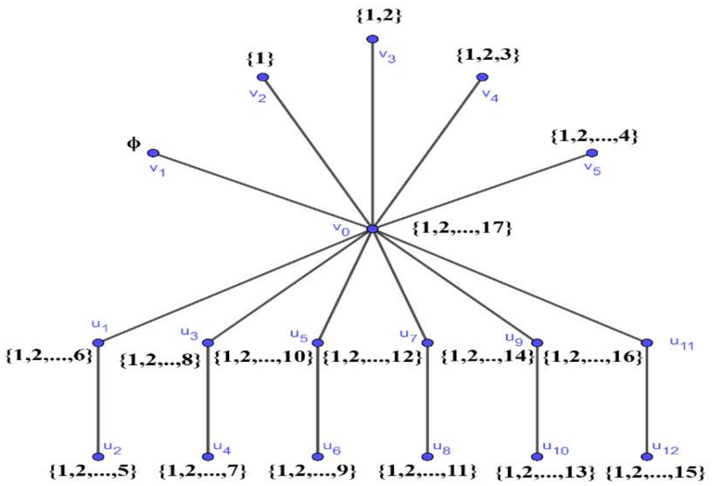

Fig 3.2 Bitopological harmonious labeling of $SP(1^5 2^6)$

Theorem 3.5

Lilly graph $L_n, n \ge 2$ is a bitopological harmonious graph.

Proof:

Let $G = L_n$.

Let $V(G) = \{u_i / 1 \le i \le 2n-1\} \cup \{v_i / 1 \le i \le 2n\}$.

Let $E(G) = \{v_i u_n / 1 \le i \le 2n\} \cup \{u_i u_{i+1} / 1 \le i \le 2n-2\}$.

$|V(G)| = 4n-1, |E(G)| = 4n-2$.

Let $X = \{1,2,\ldots,|V(G)|-1\}$.

Define a function $f: V(G) \rightarrow 2^X$ as follows:

$f(v_1) = \phi$;

$f(v_i) = \{1,2,\ldots,i-1\}$ for $2 \le i \le 2n$;

$f(u_i) = \{1,2,\ldots,2n+i-1\}$ for $1 \le i \le 2n-1$.

Here all the vertex labels are distinct and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ is given as follows:

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

Here $f^*(v_i u_n) = f(v_i)$ for $1 \leq i \leq 2n$;

$f^*(u_i u_{i+1}) = f(u_i)$ for $1 \leq i \leq 2n - 1$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.6

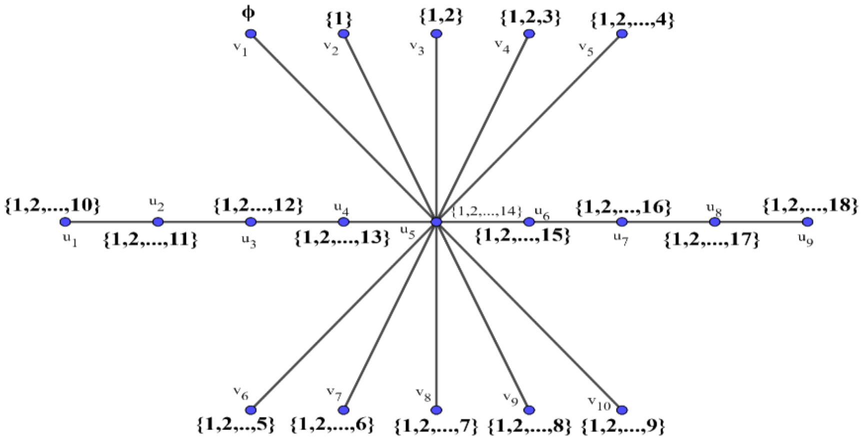

Fig 3.3 Bitopological harmonious labeling of L_5

Theorem 3.7

The firecracker graph $F_{n,k}, n, k \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = F_{n,k}$.

Let $V(G) = \{v_{ij} / 1 \leq i \leq n, 1 \leq j \leq k\}$.

Let $E(G) = \{v_{i1} v_{ij} / 1 \leq i \leq n,\ 2 \leq j \leq k\} \cup \{v_{ik} v_{i+1k} / 1 \leq i \leq n - 1\}$.

$|V(G)| = nk, |E(G)| = nk - 1$.

Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ as follows:

$f(v_{12}) = \phi$;

$f(v_{1j}) = \{1,2, \dots, j - 2\}$ for $3 \leq j \leq k - 1$;

$f(v_{i1}) = \{1,2, \dots, ki - 2\}$ for $1 \leq i \leq n$;

$f(v_{ik}) = \{1,2, \dots, ki - 1\}$ for $1 \leq i \leq n$;

$f(v_{ij}) = \{1,2, \dots, k(i - 1) + j - 2\}$ for $2 \leq i \leq n,\ 2 \leq j \leq k - 1$.

Here all the vertex labels are distinct and they form a topology on X.

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Then the induced function $f^*: E(G) \to 2^{X^*}$ is given as follows:

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

Here $f^*(v_{i1}v_{ij}) = f(v_{ij})$ for $1 \le i \le n,\ 2 \le j \le k$;

$f^*(v_{ik}v_{i+1k}) = f(v_{ik})$ for $1 \le i \le n-1$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.8

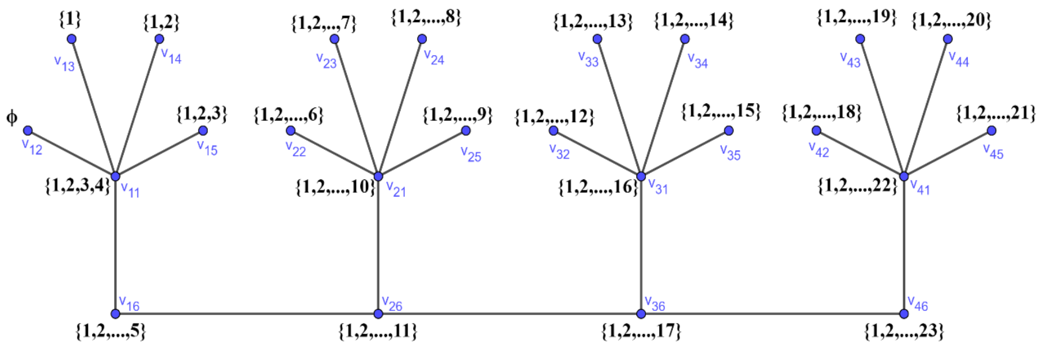

Fig 3.4 Bitopological harmonious labeling of $F_{4,6}$

CONCLUSION

In this paper, we proved some star related graphs bistar, spider graph, lilly graph and firecracker graph are bitopological harmonious graph.

REFERENCES

1. Acharya B.D., Set valuations and their applications, MRI Lecture note in Applied Mathematics, No.2, Mehta Research Institute of Mathematics and Mathematical Physics, 1983.
2. Bondy J.A and Murthy U.S.R, “Graph Theory and Application” (North Holland). New York (1976).
3. Joseph A Gallian 2018, ‘A Dynamic Survey of Graph Labeling’, The Electronic Journal of Combinatorics.
4. Selestin Lina S, Asha S, ‘On Topological Cordial Graphs’, Journal of Science and Technology, 5(2020), 25-28.
5. Selestin Lina S, Asha S, 'Topological cordial labeling of some graphs’, Malaya Journal of Matematik, Vol. 9, No. 1, 861-863.
6. Selestin Lina, S. & Asha, S. (2022), ‘Bitopological labeling of tree related graphs’, AIP Conference Proceedings. 2385. 130016. 10.1063/5.0070851.
7. G. Siva Prijith, M. Subbulakshmi, S. Chandrakala, ‘Topological Cordial Labelling of Some Graphs’, Mapana – Journal of Sciences 2023, Vol. 22, Special Issue 1, 129-140 ISSN 0975-3303.