Differential Game with Integral Constraints in a Hilbert Space

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ABSTRACT

We study a pursuit differential game problem involving one pursuer and one evader in a Hilbert space. The motion of the pursuer is governed by first-order differential equations, while that of the evader is described by a second-order differential equation. The control functions of the pursuer and the evader are subject to integral constraints. The pursuit is considered completed if the equality $y(\theta) = x(\theta)$ is achieved. We formulate and prove two theorems that provide sufficient conditions for the completion of pursuit at a given time.

Keywords: Pursuer, Integral constraint; Differential Game

INTRODUCTION

A pursuit differential game problem involves finding conditions that guarantee the completion of the pursuit. There are numerous works that deal with pursuit problems, among which are studies involving one pursuer and one evader, multiple pursuers and one evader, and even multiple evaders. For many years, differential game problems with integral constraints on the control of the players have been a subject of interest (see [2], [3], [5], [6], [7], [9], [11], [12], [15]). In some works, mixed constraints on the control of players are considered (see [1], [8], [10], [13]). In some of these papers, the motion of the players obeys first-order differential equations (e.g., [6], [9], [11]), whereas in others ([1], [2], [3]), the motion of the players obeys first- and second-order differential equations.

In [9], a pursuit differential game problem with a finite number of pursuers and one evader in the space l_2 was studied. The players move according to first-order differential equations, and the players' control functions are subject to integral constraints. The authors formulated and proved theorems that establish conditions ensuring victory for the pursuers.

A pursuit differential game problem with integral and geometric constraints was investigated in [8]. Sufficient conditions for the completion of pursuit were presented in two distinct theorems. Moreover, attainable domains and strategies for the players were also constructed.

In [1], a simple-motion pursuit differential game involving many pursuers and one evader in the Hilbert space l_2 was considered. The control functions of the pursuers and evader are subject to integral and geometric constraints, respectively. The pursuers' motions are described by first-order differential equations, while the evader's motion follows a second-order differential equation. The authors constructed strategies for the pursuers and derived a condition for the completion of pursuit.

In this paper, we study a pursuit differential game problem involving one pursuer and one evader in a Hilbert space. The motion of the pursuer is governed by a first-order differential equation, and the evader's motion is described by a second-order differential equation. The control functions of the pursuer and evader are subject to integral constraints.

Statement of the Problem

Consider the Hilbert space

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$$l_2 = \left\{ a = (a_1, a_2, a_3, \dots): \sum_{k=1}^{\infty} a_k^2 < \infty \right\},$$

with the inner product and norm defined as follows:

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k b_k, \|a\| = \left(\sum_{k=1}^{\infty} a_k^2 \right)^{\frac{1}{2}},$$

We define a pursuit differential game problem where motions of the pursuer P and that of an evader E are described by the equation

$$P: \dot{x} = u(t), \;\; x(0) = x_0 \; i \in I \quad (1)$$
$$E: \ddot{y} = v(t), \;\; \dot{y}(0) = y^1, y(0) = y^0$$

Where $x, y, u, v \in l_2$. Admissible controls satisfy:

$$\int_0^{\infty} \|u(s)\|^2 ds \le \rho^2, \quad \int_0^{\infty} \|v(s)\|^2 ds \le \sigma^2, \quad (2)$$

Definition 1: A measurable function $u(t)$ satisfying (2) is an admissible pursuer control. Similarly, $v(t)$ satisfying the evader constraint is admissible.

Definition 2: A pursuer's strategy $U(t, x, y, v)$ is admissible if it generates unique trajectories $x(t),\; y(t) \in C(0, \theta; l_2)$, for any evader's control $v(t)$, with U satisfying (2).

Definition 2: Pursuit is completed if there exist and admissible strategy U such that $x(\theta) = y(\theta)$.

THE MAIN RESULT

It is customary that (see, for example [1],[3] [10],[12] and [15]) the initial value problem involving the second order differential equation in (1) can be transformed to;

$$P: \dot{x} = u(t), \;\; x(0) = x_0 \quad (3)$$
$$E: \dot{y} = (\theta - t)v(t), \;\; , y(0) = y_0$$

With solution

$$x(\theta) = x_0 + \int_0^{\theta} u(t) dt$$

$$y(\theta) = y_0 + \int_0^{\theta} (\theta - t) v(t) dt$$

The following theorems establish sufficient conditions under which pursuit can be completed in the system defined by (1) and (2).

Theorem 1: If $x_0 = y_0$ and $\theta = \frac{\rho}{\sigma}$ then pursuit can be completed in the game described by (1), where controls u and v satisfy the inequalities described by (2)

Proof:

For the purpose of the proof of this theorem, we define the strategy of pursuer's as:

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$$U(t) = \begin{cases} (\theta - t)v(t), & t \in [0, \theta] \\ 0, & t \in (0, \infty) \end{cases} \qquad (4)$$

Admissibility (4) follows from:

$$\int_0^{\theta} \|U(t)\|^2 dt = \int_0^{\theta} (\theta - t)^2 \|v(t)\|^2 dt$$

Now, using the inequality $(\theta - t)^2 \leq \theta^2 \quad \forall t \in [0, \theta]$ to bound the integral yields

$$\int_0^{\theta} (\theta - t)^2 \|v(t)\|^2 dt \leq \theta^2 \int_0^{\theta} \|v(t)\|^2 dt$$

And the by evader's integral constraint we achieve:

$$\theta^2 \int_0^{\theta} \|v(t)\|^2 dt \leq \theta^2 \sigma^2$$

Thus,

$$\int_0^{\theta} \|v(t)\|^2 dt \leq \theta^2 \sigma^2 = \rho^2$$

Equation $x(\theta)$ and $y(\theta)$ confirms completion of pursuit i.e.

$$x(\theta) = x_0 + \int_0^{\theta} (\theta - t)v(t)\, dt = y_0 + \int_0^{\theta} (\theta - t)v(t)\, dt = y(\theta)$$

Theorem 1: If $x_0 \neq y_0$, $\rho > \left(\sigma\theta + \frac{\|y_0 - x_0\|}{\sqrt{\theta}}\right)$ and θ solves: $\rho = \frac{\|y_0 - x_0\|}{\sqrt{\theta}} + \sigma\theta$ then pursuit can be completed in the game described by (1), where controls u and v satisfy the inequalities described by (2)

Proof:

Define the strategy of the pursuer as follows:

$$U(t) = \begin{cases} \dfrac{y_0 - x_0}{\theta} + (\theta - t)v(t), & t \in [0, \theta] \\ 0, & t \in (0, \infty) \end{cases} \qquad (5)$$

By this strategy, we show that pursuit can be completed.

Hence,

$$x(\theta) = x_0 + \int_0^{\theta} U(t)\, dt$$

$$= x_0 + \int_0^{\theta} \left(\frac{y_0 - x_0}{\theta} + (\theta - t)v(t)\right) dt$$

$$= x_0 + \int_0^{\theta} \frac{y_0 - x_0}{\theta} dt + \int_0^{\theta} (\theta - t)v(t)\, dt$$

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$$= x_0 + \frac{y_0 - x_0}{\theta}\int_0^\theta dt + \int_0^\theta (\theta - t)v(t)\,dt$$

$$= x_0 + \frac{y_0 - x_0}{\theta}\cdot\theta + \int_0^\theta (\theta - t)v(t)\,dt$$

$$= x_0 + \ y_0 - x_0 + \int_0^\theta (\theta - t)v(t)\,dt$$

$$= \ y_0 + \int_0^\theta (\theta - t)v(t)\,dt = y(\theta)$$

Now, we establish the admissibility of the constructed strategy:

$$\int_0^\theta \|U(t)\|^2\,dt \le 2\left(\frac{\|\ y_0 - x_0\|^2}{\theta} + \sigma^2\theta^2\right) \le \rho^2$$

That is,

$$\|U(t)\|^2 = \left\|\frac{y_0 - x_0}{\theta} + (\theta - t)v(t)\right\|^2 \le 2\left\|\frac{y_0 - x_0}{\theta}\right\|^2 + 2(\theta - t)^2\|v(t)\|^2$$

Thus,

$$\int_0^\theta \|U(t)\|^2\,dt \le 2\int_0^\theta \left\|\frac{y_0 - x_0}{\theta}\right\|^2 + 2\int_0^\theta (\theta - t)^2\,\|v(t)\|^2 dt.$$

$$= 2\frac{\|\ y_0 - x_0\|^2}{\theta} + 2\int_0^\theta (\theta - t)^2\,\|v(t)\|^2 dt.$$

Using the evaders constraint and the bound $(\theta - t)^2 \le \theta^2\ \forall t \in [0, \theta]$, yields

$$\int_0^\theta \|U(t)\|^2\,dt \le 2\left(\frac{\|\ y_0 - x_0\|^2}{\theta} + \theta^2\sigma^2\right)$$

Relating this to the pursuer's constraint, we've

$$\int_0^\theta \|U(t)\|^2\,dt \le 2\left(\frac{\|\ y_0 - x_0\|^2}{\theta} + \theta^2\sigma^2\right) \le \rho^2$$

Which complete the prove of the theorem

Illustrative Examples

For **Theorem 1**, we require that the initial position of the pursuer and that of the evader to coincide and choose $\rho = 2, \sigma = 3, \theta = 3$. Thus (4), becomes

$$U(t) = \begin{cases} (3 - t)v(t), & t \in [0,3] \\ 0, & t \in (3, \infty) \end{cases}$$

Admissibility:

$$\int_0^\theta (3 - t)^2\|v(t)\|^2\,dt \le 9\cdot 9 = 81 = \rho^2$$

For **Theorem 2**, we require that the initial position of the pursuer and that of the evader to differ i.e. $x_0 = 0, y_0 = (1, 0, \dots), \sigma = 3, \rho = 10$

Solving $10 = \frac{1}{\sqrt{\theta}} + 3\theta$ gives $\theta \approx 3$ and the strategy's admissibility is confirmed similarly.

CONCLUSION

We studied a differential game problem involving one pursuer and one evader in a Hilbert space. The control functions of the pursuer and the evader are subject to integral constraints. The motions of the pursuer and evader are governed by first-order and second-order differential equations, respectively. We solved the pursuit problem through two theorems by constructing an admissible pursuer strategy. Consequently, we provide an illustrative example to demonstrate the obtained results

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