Numerical Solution of linear Second Order Partial Differential Equation

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ABSTRACT

This study explores the analytical and numerical solutions of partial differential equations (PDEs), focusing on parabolic (heat). The first part presents their analytical solutions using initial and boundary conditions and delves into the finite difference method (FDM), discussing forward, backward, and central difference schemes. These methods are applied to numerically solve one- and two-dimensional heat. The Crank-Nicolson method, recognized for its unconditional stability, is employed to improve the accuracy of heat equation solutions, overcoming limitations of explicit and implicit schemes. We then analyze the performance, strengths, and weaknesses of FDM through numerical simulations of one-dimensional heat. Due to computational constraints, Crank-Nicolson for 1D simulation, was not executed. Results indicate that the implicit backward difference method demonstrates superior stability by allowing unrestricted step sizes compared to the explicit forward difference method. These findings contribute to a deeper understanding of numerical PDE solutions and stability considerations in computational mathematics.

Keywords: PDEs, parabolic (heat) Equation, Crank-Nicolson Method

Mathematics Subject Classification: xx, xx

INTRODUCTION

Many problems in physical phenomena, such as physics, applied science, and engineering, can be modelled mathematically with the help of techniques of partial differential equations (John H and Fink, 1992). When a function comprises two or more independent variables, the differential equation is aptly called partial differential equation. Since the functions of multitudinous variables are inherently more problematic compared those of one variable, partial differential equations may lead to some challenging tasks in numerical problems. In fact, finding numerical solutions to those problems requires a type of scientific calculation which needs the help of a computing system (Cheney and Kincaid, 2004).

Numerous physical phenomena, which include electrostatic problems, heat conduction, fluid dynamics, electrodynamics, gravitational potential, can be modelled mathematically using partial differential equations (or PDEs) with a set of initial conditions or initial boundary conditions (Everstine, 2010). PDEs form the solid bedrock of many mathematical models related to chemical, physical, and biological phenomena, and more recently their application has diffused into the fields of financial forecasting, economics, image processing and others. (Morton and Meyers, 2005). The partial differential equation depends on two or more independent variables. These variables can be time and one or more coordinates in space or plane (Erwin, 1976).

In the study of PDE’s such as heat and wave equations, there are particular kinds of boundary condition commonly associated with above equations. For the heat equation in term of boundary condition, the initial value of the solutions are defined but in a bounded domain. The Dirichlet condition on the boundary of the domain takes positive time (t). However, the classical boundary problem in the case of the wave is equation is the Cauchy problem because it defines both the initial position and initial velocity at $t = 0$.

The stand for saying whether a boundary condition is appropriate for a particular PDE's is physically difficult to understand. However, it can be explained by fundamental mathematical insight (Hadamard, 1923 as cited in Brezis and Browder, 1997).

The onset of finite difference techniques in numerical application began in in the early 1950s and the emergence of computers that presented a convenient framework for dealing with complex problems of science and technology gave impetus to their development. The theoretical result has been found during the last five decades in term of the accuracy, consistency, stability and convergence of the finite difference method for partial differential equations (Fadugba and Adegboyegun, 2013). Furthermore, finite difference methods can be used to solve partial differential equations. This is done by approximating the differential equations over the area of integration by a system of algebraic equations. They also add that the finite difference approximations happened to be one of the simplest and oldest to solve partial differential equations. It was known by L. Euler since (1707-1783) ca. 1768, in one-dimensional space and was probably extended to two-dimensional space by C. Runge in the years (1858-1927) ca. 1908.

MATERIALS AND METHODS

Formulation of Finite Difference

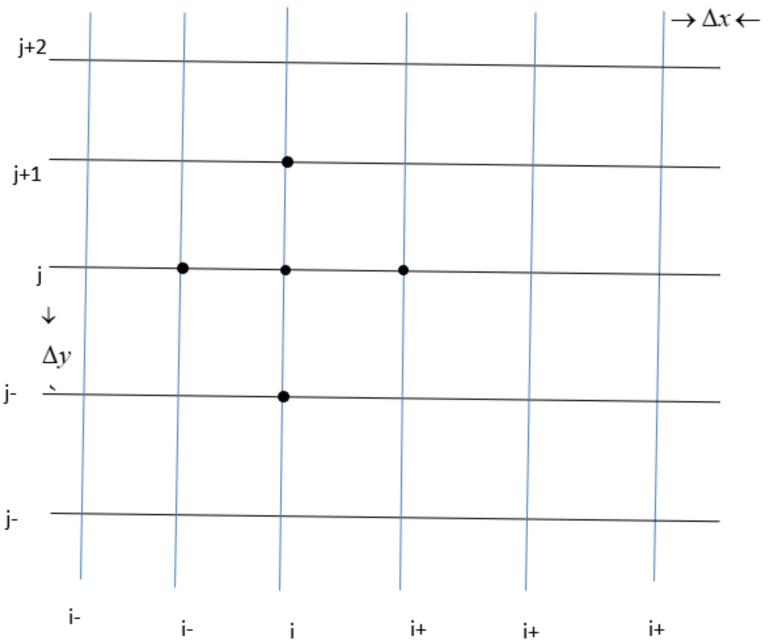

Figure 1.1

The FDM basically depends on Taylors's theorem which states that: A given function $u(x)$ has a continuous derivative over some intervals and $h = \Delta x =$ equal spacing of grid work in the x direction as in the fig. 2.1 (Curtis and Patrick, 1994), then it follows that:

$$u(x_0 + h) = u_x(x_0) + h u_x(x_0) + h^2 \frac{u_{xx}(x_0)}{2!} + \ldots + h^{n-1} \frac{u_{(n-1)}(x_0)}{(n-1)!} + O(h^n), \qquad (1)$$

$x_0 < \xi_1 < x_0 + h,$

$$u(x_0 - h) = u(x_0) - u_x(x_0)h + \frac{u_{xx}(x_0)}{2!} h^2 - \ldots + \frac{u_{(n-1)}(x_0)}{(n-1)!} h^{(n-1)} + O(h^n) \qquad (2)$$

$x_0 - h < \xi_2 < x_0 .$

Where:

(a) $u_x = \frac{du}{dx}, u_{xx} = \frac{d^2u}{dx^2}, \ldots, u_{(n-1)} = \frac{d^{n-1}u}{dx^{n-1}}$.

(b) $u_x(x_0)$ is the derivative of u with respect to x evaluated at $x = x_0$.

(c) $O(h^n)$ is an unknown error term defined.

Since $h = \Delta x$ then equations (2.1a) and (2.1b) become;

$$u(x_0 + \Delta x) = u_x(x_0) + \Delta x u_x(x_0) + \Delta x^2 \frac{u_{xx}(x_0)}{2!} + \ldots + \Delta x^{n-1} \frac{u_{(n-1)}(x_0)}{(n-1)!} + O(\Delta x^n) \quad (3)$$

Taylor Series and Finite Difference

Taylor series have been used to study the behaviour of numerical approximation to differential equations. Let us investigate the forward difference with Taylor series. To do so, we expand the function u at x_{i+1} about the point x_i (Miskandarani, 2016).

$$u(x_i + \Delta x_i) = u(x_i) + \Delta x_i \left.\frac{\partial u}{\partial x}\right|_{x_i} + \frac{\Delta x_i^{\ 2}}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \ldots \quad (3)$$

The Taylor series can rearrange to look as:

$$\frac{u(x_i + \Delta x_i) - u(x_i)}{\Delta x_i} - \left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{\Delta x_i^{\ 2}}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \ldots \quad (4)$$

It is now obvious that the forward difference formula in (4) corresponds to truncating the Taylor series after the second expression. The right hand side of equation (5) is called truncation error, because it is committed in terminating the series. The truncation error can be defined as the difference between partial derivative and its finite difference representation. The notation “Big Oh” will be used to refer to truncation error so that $T.E \approx O(\Delta x_i)$. Consequently, we can write:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_{i+1} - u_i}{\Delta x_i} + O(\Delta x_i) \quad (6)$$

The Taylor series expansion can be used obtain an expression for the truncation error corresponding to the backward difference formula:

$$u(x_i - \Delta x_{i-1}) = u(x_i) - \Delta x_{i-1} \left.\frac{\partial u}{\partial x}\right|_{x_i} + \frac{\Delta x_{i-1}^{\ 2}}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \ldots \quad (7)$$

Where $\Delta x_{i-1} = x_i - x_{i-1}$. We can obtain an expression for the error corresponding to backward difference approximation of the first derivative:

$$\frac{u(x_i) - u(x_i - \Delta x_{i-1})}{\Delta x_{i-1}} - \left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{\Delta x_{i-1}}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \ldots \quad (8)$$

The truncation error of the backward difference is not the same as the forward difference. Though, their behaviour remains similar in terms of magnitude analysis and is linear in Δx_i.

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_i - u_{i-1}}{\Delta x_i} + O(\Delta x_i) \qquad (9)$$

Notice that in both cases we have used the information provided at just two points to derive the approximation, and the error behaves linearly in both instances. Multiplying the first by Δx_{i-1} and the second by Δx_i and adding both equations we have:

$$\frac{1}{\Delta x + \Delta x_{i-1}}\left[\Delta x_{i-1}\frac{u_{i+1} - u_i}{\Delta x_i} + \Delta x_i \frac{u_i - u_{i-1}}{\Delta x_{i-1}}\right] - \left.\frac{\partial u}{\partial x}\right|_x = \frac{\Delta x_{i-1}\,\Delta x_i}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_x + \ldots \qquad (10)$$

There are several points to note on the previous expression. One the approximation uses information about the functions u at three points like: x_{i-1}, x_i and Δx_{i+1}. Two the truncation error is $T.E \approx O(\Delta x_i)$ and second order, meaning if the grid spacing is decreased by ½, the T.E. error too decreases by 2^2. Three, the preceding point can be made clearer on the important case where the grid spacing is constant: $\Delta x_{i-1} = \Delta x_i = \Delta x$. the expression simplified to:

$$\frac{u_{i+1} - u_{i-1}}{2\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x = \frac{\Delta x^3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_x + \ldots$$

Hence for an equally spaced grid the central difference approximation converges in quadratic form as $\Delta x \to 0$:

$$\left.\frac{\partial u}{\partial x}\right|_x = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2) \qquad (11)$$

Note that in equation (11), the central difference uses information at only two points. But it delivers twice the order of the other two methods. This property will generally hold whenever the grid spacing is constant and the set of points used in approximating the derivative, is symmetric (i.e. computational stencil).

Parabolic Equation 1D

Consider the finite difference method as a technique of approximating 1-dimensional heat transfer

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2}\right), \qquad 0 < x < 1, t > 0. \qquad (12)$$

The FD approximation for partial derivatives in the heat equation may lead to implicit, explicit, and crank Nicolson methods for either stability or instability, or divergence or slow convergence. But boundary conditions can determine the specific FD approximations (Suer, 2014).

Explicit Method Forward Difference

To discretize the heat equation in one dimensional from equation (12) we approximate the derivatives in x and y directions. By using grid work, as in the fig (2.1) above, to apply the central difference for the second derivative to the x yields:

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}(x,t) = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \qquad (13a)$$

If $h^2 = \Delta x^2$, then:

$$= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \qquad (13b)$$

With error $h^2 u_{xxx}(c_1,t)/12$, and forward difference formula for the derivative used for the time variable gives:

$$\frac{\partial u}{\partial t} = u_t(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \qquad (14a)$$

And if $k_d = \Delta t$ then:

$$= \frac{u(x,t+k) - u(x,t)}{k_d} \qquad (14b)$$

With error $k u_{tt}(x,c_2)/2$, where $x-h < c_1 < x+h$ and $t < c_2 < t+k$. Substituting the equations (13b) and (14b) into the equation $\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2}\right)$, yields:

$$\frac{u(x,t+k) - u(x,t)}{k} = k_d \, \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \qquad (16)$$

To employ the notation of the grid $v_{i,j} = v(i,j) = u(x_i, t_j)$, the difference equation will yield:

$$\frac{v(i,j+1) - v(i,j)}{k} = k_d \, \frac{v(i+1,j) - 2v(i,j) + v(i-1,j)}{h^2} \qquad (17a)$$

By rearranging we have

$$v(i,j+1) = v(i,j) + \frac{k k_d}{h^2}\left(v(i+1,j) - 2v(i,j) + v(i-1,j)\right) \qquad (17b)$$

Substituting $\frac{k k_d}{h^2} = \gamma$ we have:

$$v(i,j+1) = \gamma\, v(i+1,j) + (1-2\gamma)v(i,j) + \gamma v(i-1,j) \qquad (17c)$$

Note that $h = \frac{b-a}{m}$ and $k = T/N$ both in x and t as number of step and number of size respectively.

The matrix form is:

$$\begin{bmatrix} v(1,j+1) \\ v(2,j+1) \\ . \\ . \\ . \\ v(m,j+1) \end{bmatrix} = \begin{bmatrix} 1-2\gamma & \gamma & 0 & 0 & 0 \\ \gamma & 1-2\gamma & 0 & 0 & 0 \\ . & . & . & . & \gamma \\ . & & . & . & . \\ . & & & & \\ 0 & . \; . \; 0 & & \gamma & 1-2\gamma \end{bmatrix} \begin{bmatrix} v(1,j) \\ v(2,j) \\ . \\ . \\ . \\ v(m,j) \end{bmatrix} + \gamma \begin{bmatrix} v(0,j) \\ v(1,j) \\ . \\ . \\ . \\ v(m+1,j) \end{bmatrix} \qquad (18)$$

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The local truncation errors are $O(k)$ and $O(h^2)$ which is first order accurate in time and second order accurate in space. This is because they are the ones given the clear picture of the total error, as long as the method is stable. However, the initial and boundary conditions are known quantities $v(i,0)$ for $i=0,1,\ldots,M$ and $v(0,j)$ and $v(m,j)$ for $j=0,1,\ldots,N$ corresponding to the bottom of rectangle in fig (1.1)

Implicit Backward Difference

Here, the same procedure is taken but, we replace the approximation of u_t applied in the above derivation with the backward difference, so as to obtain linear implicit.

$$u_t(x,t)=\frac{u(x,t)-u(x,t-\Delta t)}{\Delta t} \quad (19)$$

Then, the equation (3.2) becomes:

$$\frac{u(x,t)-u(x,t-k)}{k}=k_d\,\frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} \quad (20a)$$

Where $\Delta t=k$ and $\Delta x^2=h^2$ with the error term $ku_{tt}(x,c_0)/2$, where $t-k<c_0$, to approximate. This is done to improve the stability of the explicit method by implicit backward difference method.

Substituting the discrete points as in equation (3.5a) yield:

$$\frac{v(i,j)-v(i,j-1)}{k}=k_d\,\frac{v(i+1,j)-2v(i,j)+v(i-1,j)}{h^2} \quad (20b)$$

By rearranging the equation (3.7b) we have:

$$v(i,j)-v(i,j-1)=\frac{kk_d}{h^2}\left(v(i+1,j)-2v(i,j)+v(i-1,j)\right) \quad (20c)$$

Taking $\frac{kk_d}{h^2}=\gamma$ and put inti equation (3.7c) will give:

$$v(i,j)-v(i,j-1)=\gamma\left(v(i+1,j)-2v(i,j)+v(i-1,j)\right) \quad (20d)$$

The final rearrangement yields:

$$-\gamma v(i+1,j)+(1+2\gamma)v(i,j)-\gamma v(i-1,j)=v(i,j-1) \quad (21)$$

The $m\times m$ matrix form of equation (21) can be written as:

$$\begin{bmatrix} 1+2\gamma & -\gamma & 0 & 0 & 0 \\ -\gamma & 1+2\gamma & 0 & 0 & 0 \\ . & . & . & . & -\gamma \\ . & & . & & \\ . & & & & \\ 0 & 0 & 0 & -\gamma & 1+2\gamma \end{bmatrix} \begin{bmatrix} v(1,j) \\ v(2,j) \\ . \\ . \\ . \\ v(m,j) \end{bmatrix} = \begin{bmatrix} v(1,j-1) \\ v(2,j-1) \\ . \\ . \\ . \\ v(m,j-1) \end{bmatrix} + \gamma \begin{bmatrix} v(0,j) \\ v(1,j) \\ . \\ . \\ . \\ v(m+1,j) \end{bmatrix} \quad (22)$$

Cranck-Nicolson Method

This method is unconditionally stable and also second order accurate in respect of both time and space. Previously, in the heat equation we noticed that, the explicit method is some time stable and the implicit method is all the times stable. When stable they both have error of order $O(k+h^2)$. The step size k requires being fairly small to have good accuracy.

This method is the right choice for solving heat equation because it is unconditionally stable. The method can deal with any system involving a conservation law.

This method is the combination of the explicit and implicit schemes. It is unconditionally stable with error $O(k^2)$ and $O(h^2)$, because of the increased accuracy and guaranteed stability. But the formulas are slightly more complicated to derive. Crank-Nicolson operates on the forward difference formula for the time derivative and evenly weighted combination of forward difference and backward difference approximation for the remaining equations (Suer, 2014).

Here the discretization of the Crank-Nicolson method comes for the variable t:

$$u_t(x,t)=\frac{v(i,j+1)-v(i,j)}{k} \qquad (23)$$

And u_{xx} with mixed difference

$$\frac{v(i+1,j)-2v(i,j)+v(i-1,j)}{2h^2}+\frac{v(i+1,j+1)-2v(i,j+1)+v(i-1,j+1)}{2h^2} \qquad (24)$$

Again putting $\gamma = {kk_d}/{h^2}$ and rearranging the heat equation approximation to form:

$$2v(i,j+1)-2v(i,j)=\gamma[v(i+1,j)-2(i,j)+v(i-1,j)+v(i+1,j+1)-2v(i,j+1)+v(i-1,j+1)] \qquad (25a)$$

Or in another form:

$$-\gamma v(i+1,j+1)+(2+2\gamma)v(i,j+1)-\gamma v(i-1,j+1)=v(i+1,j)+(2-2\gamma)v(i,j)+\gamma v(i-1,j) \qquad (25b)$$

We can see that, the left hand side of equation (4.2b) consists of unknown expressions while the right hand side contains known quantities. The equation (25b) forms a tridiagonal matrix.

To set this matrix, let $v_j=[v_{1j},\ldots,v_{mj}]^T$. The Crank-Nicolson method of the form

$$Bv_{j+1}+\gamma(c_{j+1}+c_j)=Av_j$$

where

$$B=\begin{bmatrix} 1+2\gamma & -\gamma & 0 & 0 & 0 \\ -\gamma & 1+2\gamma & 0 & 0 & 0 \\ . & . & . & . & -\gamma \\ . & & & . & \\ . & & & & \\ 0 & 0 & 0 & -\gamma & 1+2\gamma \end{bmatrix}, \quad A=\begin{bmatrix} 1-2\gamma & \gamma & 0 & 0 & 0 \\ \gamma & 1-2\gamma & 0 & 0 & 0 \\ . & . & . & . & \gamma \\ . & & . & . & . \\ . & & & & \\ 0 & . & . & 0 & \gamma & 1-2\gamma \end{bmatrix}, \qquad (26)$$

And $s_j = \left[v_{0,j}, 0\ldots, v_{m+1,j}\right]^T$

See the below grid:

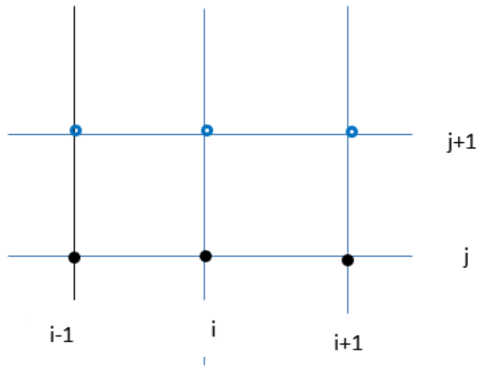


Figure 1.2

Figure 1.2 mesh point for Crank-Nicolson method. The open circles are the unknowns at each time step and the filled circles are known representing the initial and boundary conditions.

RESULTS

This section has solved example of heat (explicit forward difference and implicit backward difference methods with an exclusion of Crank Nicolson) equations numerically only one dimension by using Finite Difference Method (FDM).

Heat Diffusion equation in one Dimension

Suppose the initial and boundary conditions for the heat equation (12) are given below:

$$u(x,0) = \sin(\pi x) = f(x) \quad 0 < x < 1 \text{ for all } t = 0 \qquad (27a)$$

$$u(0,t) = u_0 = 0$$

for $x = 0$ and $0 \le t \le 1.5$ (27b)

$$u(L,t) = u_L = 0 \quad \text{for } L = t = 0 \text{ and } 0 \le t \le 1.5 \qquad (27c)$$

In this illustration, we use the step sizes $\Delta x = h = 0.137$ $\Delta t = 0.1$ $k_d = 2$, and $k = 1$. Thus, the ratio is $\gamma = 0.5$ the number of the grid used is n= 16 columns wide and m= 12 rows high.

Table 1. Using the Forward-difference with $\gamma = 0.5$

t	x	0	0.137	0.274	0.411	0.548	0.685	0.822	0.959	1.096	1.233	1.37
0		0	0.41723	0.75836	0.96117	0.98865	0.83581	0.53051	0.12845	-0.297	-0.6684	0
0.1		0	0.37918	0.6892	0.87351	0.89849	0.75958	0.48213	0.11673	-0.27	-0.1485	0
0.2		0	0.3446	0.62634	0.79384	0.81654	0.69031	0.43816	0.10609	-0.0159	-0.135	0
0.3		0	0.31317	0.56922	0.72144	0.74208	0.62735	0.3982	0.21113	-0.0144	-0.0079	0
0.4		0	0.28461	0.51731	0.65565	0.6744	0.57014	0.41924	0.19188	0.10159	-0.0072	0
0.5		0	0.25865	0.47013	0.59585	0.61289	0.54682	0.38101	0.26042	0.09233	0.0508	0
0.6		0	0.23506	0.42725	0.54151	0.57134	0.49695	0.40362	0.23667	0.15561	0.04616	0
0.7		0	0.21363	0.38829	0.49929	0.51923	0.48748	0.36681	0.27961	0.14142	0.0778	0
0.8		0	0.19414	0.35646	0.45376	0.49339	0.44302	0.38355	0.25411	0.17871	0.07071	0
0.9		0	0.17823	0.32395	0.42492	0.44839	0.43847	0.34857	0.28113	0.16241	0.08935	0
1		0	0.16198	0.30158	0.38617	0.43169	0.39848	0.3598	0.25549	0.18524	0.0812	0
1.1		0	0.15079	0.27407	0.36664	0.39232	0.39575	0.32698	0.27252	0.16835	0.09262	0
1.2		0	0.13704	0.25871	0.3332	0.38119	0.35965	0.33413	0.24766	0.18257	0.08417	0
1.3		0	0.12936	0.23512	0.31995	0.34643	0.35766	0.30366	0.25835	0.16592	0.09128	0
1.4		0	0.11756	0.22465	0.29077	0.33881	0.32504	0.30801	0.23479	0.17482	0.08296	0
1.5		0	0.11233	0.20417	0.28173	0.30791	0.32341	0.27992	0.24141	0.15887	0.08741	0

Figure 1a

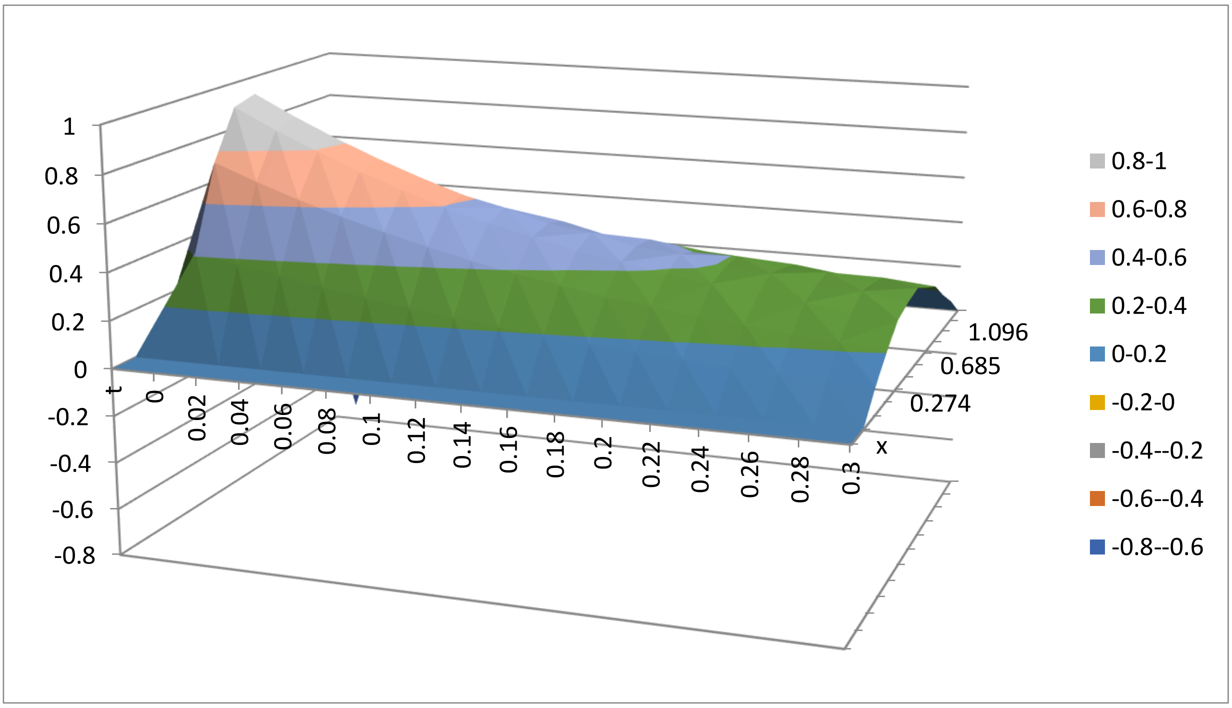
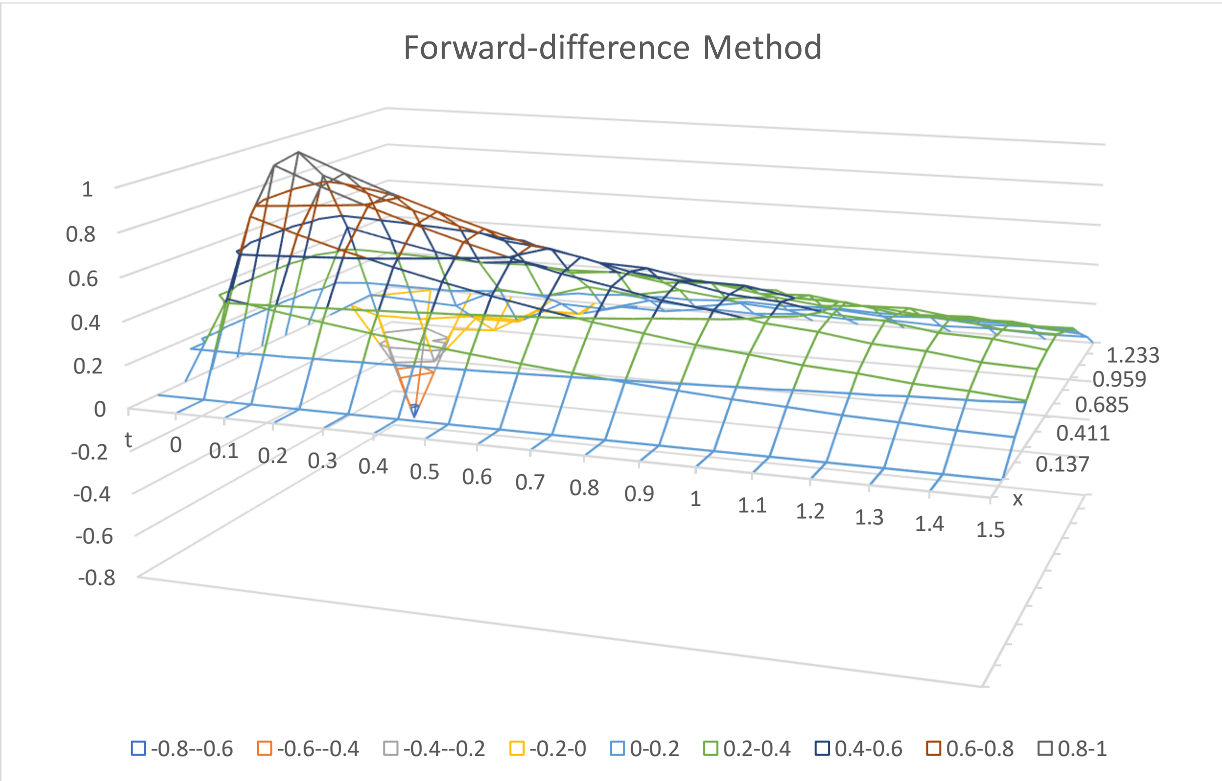


Figure 1b Using the Forward difference method with r=0.5



The formula used is in equation (17c). This FDM method is stable for $\gamma = 0.5$ which suffices its criteria and can be used to form accurate approximation to the solution $u(x,t)$ for $0 \le t \le 0.15$ and $0 \le x \le 1.37$ as given in Table 1. The consecutive rows and columns are given in Figure (1a). A three dimensional representation of data in table (1) given in figure (1a) and (1b).

In our illustration, we use the step sizes $\Delta x = h = 0.137$ $\Delta t = 0.02$, $k_d = 2$, and $k = 1$. Thus, the ratio is $\gamma = 0.5$ the number of the grid used is n= 11 columns wide and m= 16 rows high.

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Table 2. Using the Forward-difference with $\gamma = 0.55$

x	0	0.137	0.274	0.411	0.548	0.685	0.822	0.959	1.096	1.233	1.37
t											
0	0	0.41723	0.75836	0.96117	0.98865	0.83581	0.53051	0.12845	-0.297	-0.6684	0
0.1	0	0.37538	0.68228	0.86474	0.88947	0.75196	0.47729	0.11556	-0.2672	-0.0965	0
0.2	0	0.33772	0.61384	0.77799	0.80024	0.67652	0.42941	0.10397	0.03719	-0.1373	0
0.3	0	0.30384	0.55226	0.69994	0.71996	0.60865	0.38633	0.24623	-0.0221	0.03419	0
0.4	0	0.27336	0.49685	0.62972	0.64773	0.54759	0.43155	0.17572	0.15644	-0.0156	0
0.5	0	0.24593	0.44701	0.56655	0.58275	0.53885	0.35467	0.30582	0.07245	0.0876	0
0.6	0	0.22126	0.40216	0.50971	0.54969	0.4617	0.4291	0.20433	0.20913	0.03109	0
0.7	0	0.19906	0.36182	0.47255	0.47931	0.49217	0.3234	0.3306	0.10857	0.11192	0
0.8	0	0.17909	0.33321	0.41536	0.48266	0.39227	0.42018	0.20452	0.23253	0.04852	0
0.9	0	0.16535	0.29363	0.40719	0.39593	0.45734	0.28622	0.33854	0.11592	0.12304	0
1	0	0.14496	0.28554	0.33854	0.4359	0.32945	0.40911	0.18733	0.24227	0.05145	0
1.1	0	0.14255	0.23737	0.36294	0.32381	0.43181	0.24332	0.33953	0.1071	0.1281	0
1.2	0	0.1163	0.25428	0.27236	0.40473	0.26874	0.3999	0.15878	0.24649	0.0461	0
1.3	0	0.12822	0.18833	0.33522	0.25713	0.41567	0.19514	0.33964	0.08803	0.13096	0
1.4	0	0.09076	0.23606	0.21148	0.38728	0.20718	0.39591	0.12178	0.25002	0.03532	0
1.5	0	0.12076	0.14263	0.32169	0.19154	0.41003	0.14134	0.34308	0.0614	0.13398	0

Figure 2a

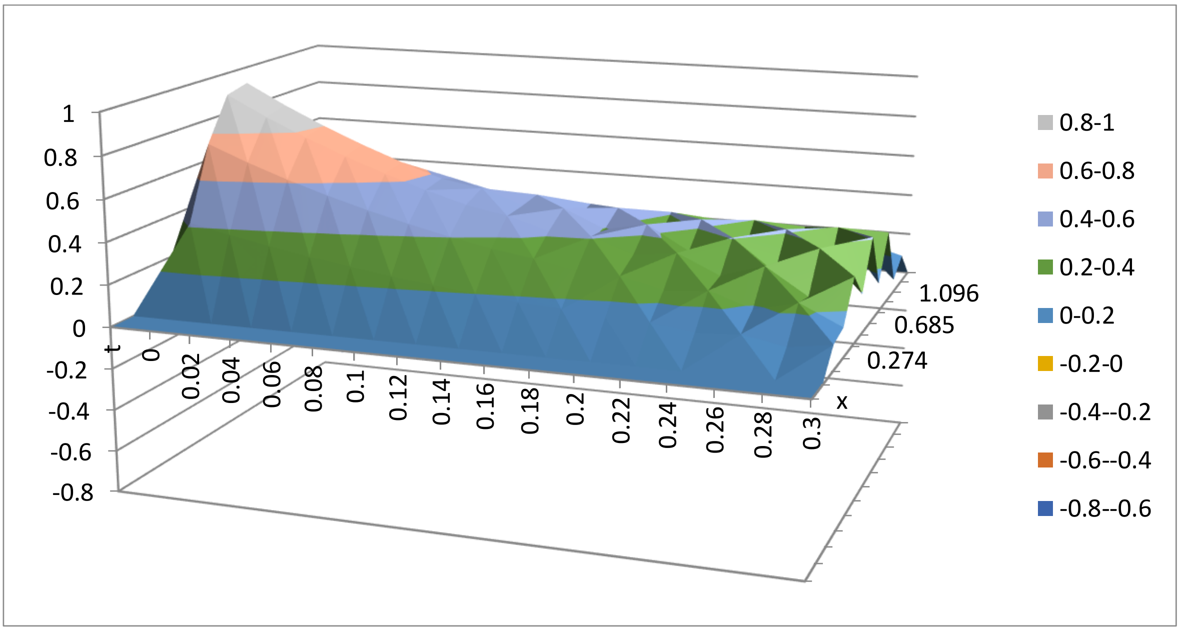
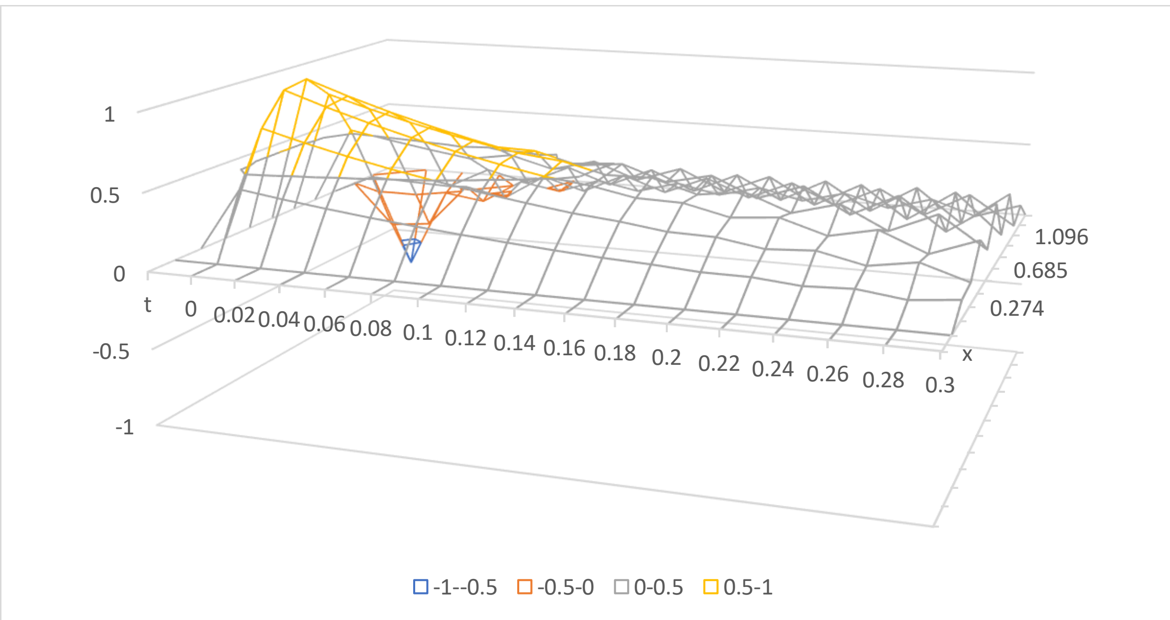


Figure 2b



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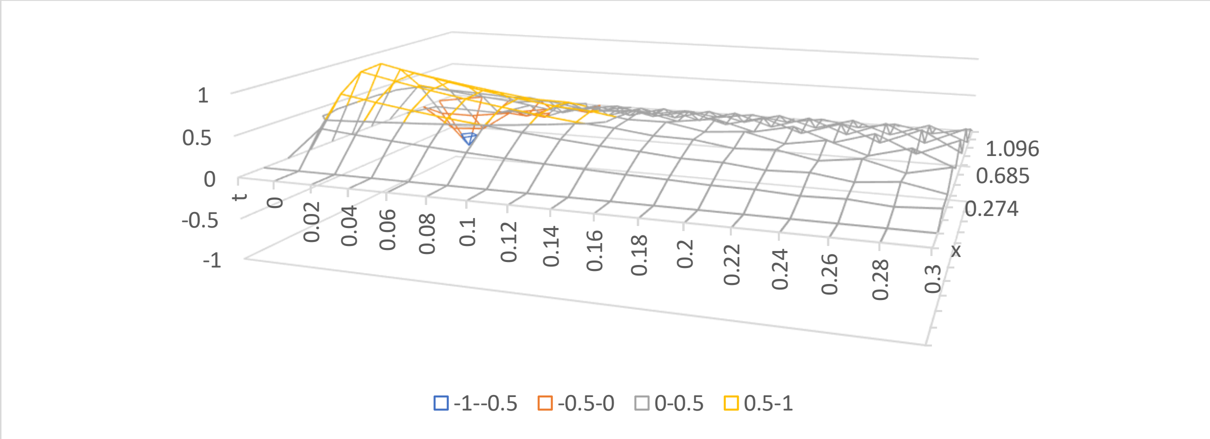


Here, the formula used in equation (17c) confirmed that the FDM with $\gamma = 0.55$ is unstable, because $\gamma > \frac{1}{2}$, which means the error occurred at one row will be magnified in successive rows. The difference equation has accuracy of order $O(k) + O(h^2)$. Because the term $O(k)$ reduces linearly as k tend to zero, which means it must be made small to produce good approximation.

In this illustration, we use the step sizes $\Delta x = h = 0.1\ \Delta t = 0.004,\ k_d = 2,$ and $k = 1$. Thus, the ratio is $\gamma = 0.5$ the number of grid used is n= 16 columns wide and m= 12 rows high.

Table 3. Using Backward difference with $\gamma = 0.69$

	x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
t												
0		0	0.309017	0.587785	0.809017	0.951057	1	0.951057	0.809017	0.587785	0.309017	0
0.004		0	0.329889	0.627485	0.86366	1.015293	1.067542	1.015293	0.86366	0.627485	0.329889	0
0.008		0	0.35217	0.669867	0.921993	1.083868	1.139646	1.083868	0.921993	0.669867	0.35217	0
0.012		0	0.375956	0.715111	0.984266	1.157074	1.21662	1.157074	0.984266	0.715111	0.375956	0
0.016		0	0.401349	0.763411	1.050745	1.235225	1.298793	1.235225	1.050745	0.763411	0.401349	0
0.02		0	0.428457	0.814974	1.121715	1.318655	1.386516	1.318655	1.121715	0.814974	0.428457	0
0.024		0	0.457396	0.870019	1.197478	1.40772	1.480164	1.40772	1.197478	0.870019	0.457396	0
0.028		0	0.488289	0.928781	1.278358	1.5028	1.580137	1.5028	1.278358	0.928781	0.488289	0
0.032		0	0.521269	0.991513	1.364701	1.604302	1.686863	1.604302	1.364701	0.991513	0.521269	0
0.036		0	0.556477	1.058482	1.456875	1.71266	1.800797	1.71266	1.456875	1.058482	0.556477	0
0.04		0	0.594062	1.129974	1.555276	1.828336	1.922426	1.828336	1.555276	1.129974	0.594062	0
0.044		0	0.634187	1.206295	1.660322	1.951826	2.052271	1.951826	1.660322	1.206295	0.634187	0
0.048		0	0.677021	1.28777	1.772464	2.083656	2.190885	2.083656	1.772464	1.28777	0.677021	0
0.052		0	0.722748	1.374749	1.892179	2.22439	2.338862	2.22439	1.892179	1.374749	0.722748	0
0.056		0	0.771564	1.467602	2.019981	2.37463	2.496834	2.37463	2.019981	1.467602	0.771564	0

Using Back ward difference with $\gamma = 0.69$

Figure 3a

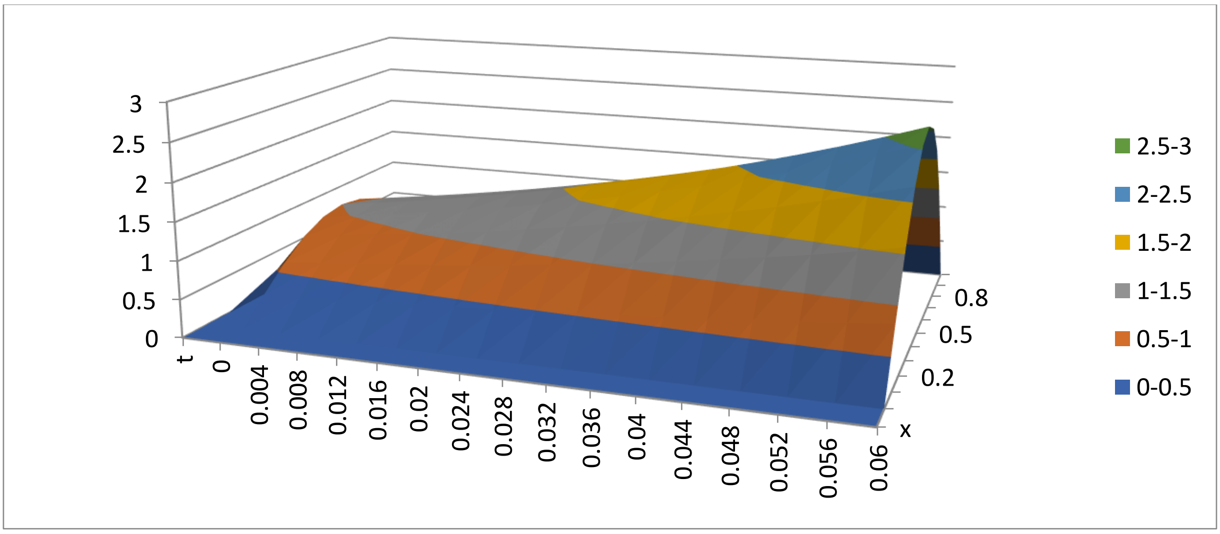
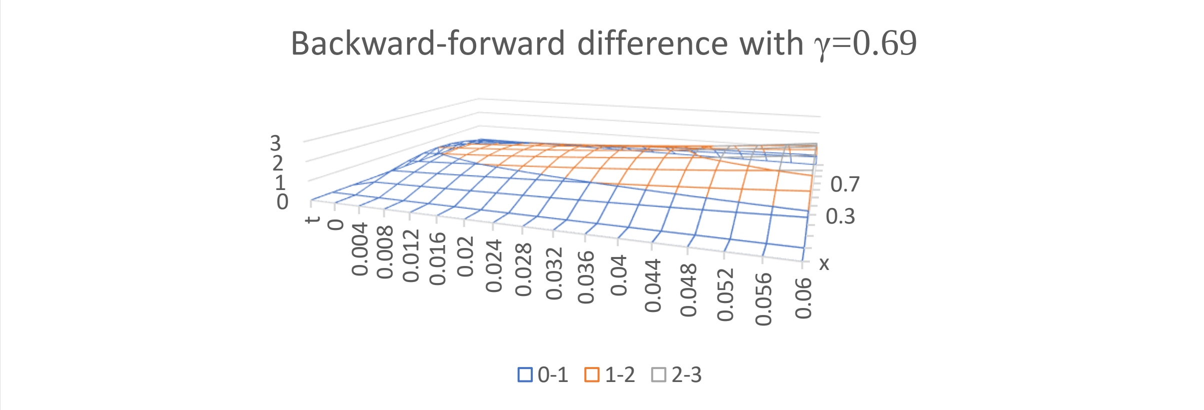


Figure 3b



The formula used is in equation (3.7d). This FDM method is stable for $\gamma = 0.69$ which satisfies its criteria of being not restricted to half (1/2) and can be used to form accurate approximation to the solution $u(x,t)$ for $0 \le t \le 0.056$ and $0 \le x \le 1$ as given in Table 3. The consecutive rows and columns are given in Figure (4.3). A three-dimensional representation of data in table (3) given in figure (3a) and (3b).

DISCUSSION

Strengths and Weaknesses Of The Finite Difference Methods

This section will give a brief account on how strength and weak finite difference methods are based on analysis of the stability of forward difference (explicit method) and its criteria of being stable and give a proof for stability criteria. Also, the stability of the backward difference (implicit method) their truncation error and Crank Nicolson method as a remedy for short comings of the first two methods.

Stability Analysis for Forward Difference (Explicit Method)

The strange behaviour occurred in the previous heat simulation gave rise to a problem. The solution to partial differential equations by the implicit forward difference method, can take a good care of error amplification or magnification for practical step size. This happened to be a crucial and pivotal aspect of stable and efficient solution. Here, the maximum value of the ratio of our step sizes (γ) is half (1/2) as in figure (1a) and (1b) but for proof see equation $\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$.. When the ratio value (γ) exceeds half (1/2), the explicit forward difference is said to be conditionally stable. The reason is its stability depends on the choice of step sizes. This method is first order accurate in time and second in space.

The discretized form of equation (17c) is one of the contributing factors for truncation error because of the approximations of the derivative of (12) and also error magnification due to the method itself. Von Newman stability analysis measures the error amplification or magnification. Looking more closely at what the finite difference method is doing help us to investigate this magnification. To have a stable method, step size must be selected in such a way that the amplification factor should not be larger than 1(Sauer T, 2014).

In an explicit scheme, the temperature at time n+1 depends explicitly on the temperature at time n.

Stability Analysis for Backward Difference (Implicit Method)

In order to improve the stability of the explicit forward difference method, we use implicit backward difference method (Euler method) in equation (20d). Even though the magnitude of the truncation error of the explicit in (17c) is similar to that of implicit of (21) with different matrix arrangement, the nature analysis of stability of the implicit backward difference is similar to explicit forward difference case. Although, the value of the ratio of our step sizes (γ) is not restricted to half (1/2) for stability which is rather bigger than that of explicit method.

Hence the implicit method is stable for all value of the ratio of step sizes (γ). The backward difference method is unconditionally stable and is first order accurate in time and second space. In term of two-dimensional heat, the scheme for implicit method provides second order convergence because only a very few iterations per time were needed.

Since this method is stable for all step size, the errors from both the time and space discretization are of the order $O(k)$ and $O(h^2)$ respectively. This implies that, for small step $k \approx h$, the error from the time step size will dominate, since $O(h^2)$ will be in negligible amount compared with $O(k)$. In other form the error can be written as $O(k)+O(h^2)=O(k)$.

Crank Nicolson Method

This method is unconditionally stable and also second order accurate in respect of both time and space. Previously, in the heat equation we noticed that, the explicit method is some time stable and the implicit method is all the times stable. When stable they both have error of order $O(k+h^2)$. The step size k requires being fairly small to have good accuracy (Sauer T, 2014).

Crank Nicolson is suitable finite difference method for the heat equation because it is unconditionally stable and second order convergence. Deriving this method cannot be achieved directly due to the first partial derivative u_t appeared in the equation (Sauer T, 2014). Although the explicit method is unconditionally stable with a serious draw back, the time step is very small.

In improving accuracy of Crank Nicolson method, one may employ higher order multistep methods or implicit Runge-Kutta methods to improve the Crank Nicolson method in time. To improve accuracy of Crank Nicolson, one can use analogy with Nomerov's method which is out of the confines of this method. $O\left(k^2+h^2+\frac{k^2}{h^2}\right)$

Observations

- The error of numerical solutions increases with number of steps.
- These errors are called Accumulative errors.
- Step size has strong effect on the accuracy on the finite difference method.
- Trade-off between computational effort and step size is an issue in any numerical technique such as FDM.

Stability of the Finite Difference method for the heat equation Consider the following approximation to the 1-D heat equation (Gordon D. Smith, 2004):

$$u_n^{k+1}-u_n^k=\frac{\Delta t}{\Delta x^2}\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right)$$

Where $u_n^k \cong u(x_n,t_k)$

According notation used throughout this work is

$$v(i,j+1)-v(i,j)=\frac{\Delta t}{\Delta x^2}\left(v(i+1,j)-2v(i,j)+v(i-1,j)\right)$$

Such that $u_n^k=v_n^k=v(i,j)$.

Let

$u_n^k = \phi k e^{in\Delta x\theta}$ then (4.1)

$$\left(\phi_{k+1} - \phi_k\right)e^{in\Delta x\theta} = \frac{\Delta t}{\Delta x^2}\left(e^{i\Delta x\theta} - 2 + e^{i\Delta x\theta}\right)e^{in\Delta x\phi} \quad (28)$$

$$= \frac{\Delta t}{\Delta x^2}\left[2\cos(\theta\Delta x) - 2\right] \quad (29)$$

According to trig identity we have that $\cos(\theta\Delta x) - 1 = -2\sin^2\left(\frac{\theta\Delta x}{z}\right)$

Therefore

$$\phi_{k+1} = \phi_k - \frac{\Delta t}{\Delta x^2}4\sin^2\left(\frac{\theta\Delta x}{z}\right)\phi_k \quad (30)$$

$$= \left[1 - \frac{\Delta t}{\Delta x^2}4\sin^2\left(\frac{\theta\Delta x}{z}\right)\right]\phi_k \quad (31)$$

Now for the sake of stability we need $\left|\phi_{k+1} - \phi_k\right| < \left|\phi_k\right|$ such that

$$\left|1 - \frac{\Delta t}{\Delta x^2}4\sin^2\left(\frac{\theta\Delta x}{z}\right)\right| < 1 \quad (32)$$

$$-2 \le \frac{\Delta t}{\Delta x^2}4\sin^2\left(\frac{\theta\Delta x}{z}\right) \le 0 \quad (33)$$

The right inequality suffices automatically while the left one can be written again in the form:

$$\frac{4\Delta t}{\Delta x^2}\sin^2\left(\frac{\theta\Delta x}{z}\right) \le 0 \quad (34)$$

Since $\sin(...) \le 1$this condition satisfies for whatever value of θ provided

$$\Delta t \le \frac{\Delta x^2}{2} \quad (35)$$

By rearranging the above we arrived at

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}. \quad (36)$$

CONCLUSIONS

This research has introduced method for solving 1-D linear second order partial differential equation for Parabolic Equation using finite difference method (FDM). The FDM with three types as the forward, central and backward differences are used to solve heat equation in 1 Dimension. The unconditionally Crank Nicolson method is applied solve heat problems in both the dimensions. Strengths and weaknesses (in term of stability and accuracy) of these methods were checked by numerical solution.

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The implicit method is more stable than explicit and the Crank Nicolson is the in term of stability and accuracy. The Crank Nicolson is unconditionally stable. In FDM, the step taking for solving is convergent and accurate. Successive over relaxation method is applied in elliptic equation to speed up the rate of convergence. When this method is applied, the number of iterations reduces drastically.

In this research, we consider numerical methods for PDEs and obtain the approximations for finite difference method.

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Conflict of interest: The authors declare no conflicts of interest.

REFERENCES

1. Abdullah, Abdul Rahman Bin. "The study of some numerical methods for solving parabolic partial differential equations." PhD diss., Loughborough University, 1983.
2. Data Engineering and Applications: Proceedings of the International Conference, IDEA 2K22, **Volume 1.** Singapore: Springer Nature Singapore, 2024.
3. De Weck, O., and I. Y. Kim. "Finite Element Method. Engineering Design and Rapid Prototyping." Massachussetts Institute of Technology (2004).
4. Dhatt, Gouri, Emmanuel Lefrançois, and Gilbert Touzot. Finite element method. John Wiley & Sons, 2012.
5. Hyman, James M., and Bernard Larrouturou. "The numerical differentiation of discrete functions using polynomial interpolation methods." Applied Mathematics and Computation **10** (1982): 487-506.
6. Johnson, Claes. Numerical solution of partial differential equations by the finite element method. Courier Corporation, 2009.
7. Liu, Gui-Rong, and Siu Sin Quek. The finite element method: a practical course. Butterworth-Heinemann, 2013.
8. Mathews, John H., and Kurtis D. Fink. Numerical methods using MATLAB. **Vol. 4**. Upper Saddle River, NJ: Pearson prentice hall, 2004.
9. Pozrikidis, C. "Finite and spectral element methods using Matlab." University of California, San Diego (2005).
10. Rao, Singiresu S. The finite element method in engineering. Elsevier, 2010.
11. Smith, Gordon D. Numerical solution of partial differential equations: finite difference methods. Oxford university press, 1985.