On Some New Sets Via Local Closure Function in Ideal Topological Space

Ali Bulama mamman, Haruna Usman Idriss, Bashir Mai Umar, Buba M.T Hambadga and Jamilu Adamu

Department of Mathematics, Federal University Gashua, Yobe State, Nigeria.

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ABSTRACT

In this paper, we introduce Lr-perfect set, Rr-perfect set and Cr-perfect set in ideal topological space and study their properties. We investigate the relationship between the existing R*-perfect sets and Rr-perfect set and also L*-perfect sets and Lr-perfect set. We construct a topology τT by using Kuratowski closure operator.

Keywords: Lr-perfect, Rr-perfect and Cr-perfect sets

INTRODUCTION

The concept of ideal in topological space was introduced by K. Kuratowski in (1930)[?] as a nonempty collection I of subsets of a topological space (X, τ) that satisfy the following conditions:

1. If $A \in I$ and $B \subseteq A$, then $B \in I$ (*heredity*)
2. If $A, B \in I$, then $A \cup B \in I$ (*finite additive*)

The space (X, τ, I) is called an ideal topological space. In 1933 Kuratowski [?] introduced the notion of local function $Q_x : P(X) \to P(X)$ defined as $A^*\_{x}(I, \tau) := \{x \in X : U \cap A \notin I\}$ for every open set U containing x and A is any subset of a topological space X. In 2013 Ahmed Al-Omari and Takashi Noiri [?] introduced the notion of local closure function $\Gamma : P(X) \to P(X)$ defined as $\Gamma(A)(I, \tau) = \{x \in X : A \cap Cl(U) \notin I$ for every $U \in \tau(x)$, where A is any subset of a topological space X. If there is no ambiguity, we use $A\backslash_x$ and $\Gamma(A)$ instead of $A^*\_{x}(I, \tau)$ and $\Gamma(A)(I, \tau)$. In 2013 R. Manoharan and P. Thangavelu [?] used local function and ideal to introduce $R\backslash_x$-perfect, $L\backslash_x$-perfect and $C\backslash_x$-perfect sets and also in 2018 Lawrence et al [?] introduced R_I-perfect, L_I-perfect and C_I-perfect set in ideal topological space. In this paper we introduce L_r-perfect, R_r-perfect and C_r-perfect sets as a generalisation to $R\backslash_x$-perfect, $L\backslash_x$-perfect and $C\backslash_x$-perfect sets respectively.

PRELIMINARIES

The following definitions, lemmas, and theorems are very important in this research.

Definition 2.1

If (X, τ, I) is an ideal topological space and A ⊆ X. Then the following hold:

1. A is τ_x-closed if $A_x \subseteq A$.
2. A is $_x$-dense-itself if $A \subseteq A_x$.
3. A is I-dense if A = X.

4. A is I-open if $A \subseteq (int(A))_x$.
5. A is regular I-closed if $A = (int(A))_x$.
6. A is almost I-open if $A \subseteq cl((int(A))_x)$.

Definition 2.2

If (X, τ, I) is an ideal topological space, then a topology τ is compatible with ideal I if for every $A \subseteq X$: if for every $x \in A$ there exist $U \in \tau(x)$ such that $U \cap A \in I$, then $A \in I$ denoted by $\tau \sim I$.

Lemma 2.3

Let (X, τ) be a topological space and I_1 and I_2 be two ideals on X. If A, B ⊆ X, then the following hold:

1. If $A \subseteq B$ then, $A_x \subseteq B_x$.
2. If $I_1 \subseteq I_2$ then, $A_x(I_2) \subseteq A_x(I_1)$.
3. $A_x = cl(A) \subseteq cl(A)$ (A_x is closed subset of cl(A)).
4. $(A_x)_x \subseteq A_x$.
5. $(A \cup B)_x = A_x \cup B_x$.
6. $A_x - B_x = (A - B)_x - B_x \subseteq (A - B)_x$.
7. For every $I_1 \in I$, $(A \cup I_1)_x = A_x = (A \cup I)_x$.

Theorem 2.4 Let (X, τ, I) be an ideal topological space. Then the following are equivalent:

1. $\tau \sim I$
2. If $A \subseteq X$ has a cover of open sets whose intersection with A belong to I.
3. If for every $A \subseteq X$, $A \cap A^*_x = \emptyset$, then $A \in I$.
4. If for every $A \subseteq X$, $A - A^*_x \in I$.
5. If for every $A \subseteq X$, if A contains a nonempty subset B with $B \subseteq B^*_x$, then $A \in I$.

Theorem 2.5 Let (X, τ, I) be an ideal topological space. Then the following are equivalent:

1. $\tau \sim_x I$
2. If $A \subseteq X$ has a cover of sg-open sets whose intersection with A belong to I.
3. If for every $A \subseteq X$, $A \cap \Gamma(A) = \emptyset$, then $A \in I$.
4. If for every $A \subseteq X$, $A - \Gamma(A) \in I$.
5. If for every $A \subseteq X$, if A contains a nonempty subset B with $B \subseteq \Gamma(B)$, then $A \in I$.

Theorem 2.6 If (X, τ, I) is an ideal topological space and A, B ⊆ X, then the following hold:

1. $\Gamma(\emptyset) = \emptyset$.

2. $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$.
3. If $A \subseteq B$, then $\Gamma(A) \subseteq \Gamma(B)$.

THE OPEN SETS OF TΓ

In this section, we investigate $\tau\Gamma$ finer than τ^*_x, called the Kuratowski local closure operator, i.e. $Cl\Gamma(A) = A \cup \Gamma(A)$. A subset of ideal topological space (X, τ, I) is said to be $\tau\Gamma$-closed if $\Gamma(A) = A$. i.e., if $U \in \tau\Gamma$, then $X - U$ is $\tau\Gamma$-closed implies $\Gamma(X - U)(X - U)$ if and only if $U \subseteq X - \Gamma(X - U)$. Therefore, $x \in U$ implies $x \notin \Gamma(X - U)$ implies there exists $V \in N(x)$ such that $V \cap (X - U) \in I$. Let $I = V \cap (X - U)$ and we have $x \in V - I \subseteq U$, which is a basis for $\tau\Gamma$ denoted by $\beta(I_1, \tau) = \{V - I_1 : V \in \tau, I_1 \in I\}$.

Theorem 3.1

Let (X, τ, I) be an ideal topological space, $A, B \subseteq X$ and $Cl\Gamma(A) = \Gamma(A) \cup A$, then the following hold:

1. $Cl\Gamma(\emptyset) = \emptyset$.
2. $A \subseteq Cl\Gamma(A)$.
3. $Cl\Gamma(A \cup B) = Cl\Gamma(A) \cup Cl\Gamma(B)$.
4. $Cl\Gamma(A) = Cl\Gamma(Cl\Gamma(A))$.

Proof:

(1) By theorem ?? $\Gamma(\emptyset) = \emptyset$. Therefore $Cl\Gamma(\emptyset) = \Gamma(\emptyset) \cup \emptyset = \emptyset$. Hence $Cl\Gamma(\emptyset) = \emptyset$.

(2) $A \subseteq A \cup \Gamma(A) = Cl\Gamma(A)$.

(3) $Cl\Gamma(A \cup B) = \Gamma(A \cup B) \cup (A \cup B) = \Gamma(A) \cup \Gamma(B) \cup (A \cup B) = Cl\Gamma(A) \cup Cl\Gamma(B)$.

(4) $Cl\Gamma(Cl\Gamma(A)) = Cl\Gamma(\Gamma(A) \cup A) = \Gamma(\Gamma(A) \cup A) \cup (\Gamma(A) \cup A) = ((\Gamma(A) \cup \Gamma(A)) \cup (\Gamma(A) \cup A) = \Gamma(A) \cup A = Cl\Gamma(A)$.

LΓ-perfect, RΓ-perfect, and CΓ-perfect Sets

Definition 4.1

Let (X, τ, I) be an ideal topological space. A subset A of the space X is said to be:

1. **LΓ-perfect** if the difference A minus $\Gamma(A)$ is in the ideal I.
2. **RΓ-perfect** if the difference $\Gamma(A)$ minus A is in the ideal I.
3. **CΓ-perfect** if the set is both LΓ-perfect and RΓ-perfect.

Lemma 4.2

Let (X, τ, I) be an ideal topological space. Then A^* is a subset of $\Gamma(A)$.

Proof:

Let x be in A^*. Then A intersected with any open set U containing x is not in I. Since A intersect U is a subset of A intersect Cl(U), and A intersect Cl(U) is not in I, we conclude that A intersect U is also not in I.

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Example:

Let X = {a, b, c},

τ (the topology) = {empty set, X, {a}, {a, b}},

I (the ideal) = {empty set, {a}}.

If A = {a, b}, then

A^* = {b} and Γ(A) = {a, b}.

Proposition 4.3

If a subset A of an ideal topological space (X, τ, I) is CΓ-perfect, then the symmetric difference of A and Γ(A) is in I.

Proof:

Since A is both L_Gamma-perfect and R_Gamma-perfect, then

A - Gamma(A) ∈ I and Gamma(A) - A ∈ I.

By the finite additive property of ideals:

(A - Gamma(A)) ∪ (Gamma(A) - A) ∈ I.

Hence, symmetric difference Delta(Gamma(A)) ∈ I.

Example:

Let X = {a, b, c},

τ = {∅, X, {b}, {a, b}},

I = {∅, {c}}.

If Gamma(A) = {a, b} and A = {a, b, c}, then:

Gamma(A) - A = ∅,

A - Gamma(A) = {c},

So, Delta(Gamma(A)) = {c} ∈ I.

Proposition 4.4:

Every τ_Gamma-closed set is R_Gamma-perfect set in an ideal topological space (X, τ, I).

Proof:

Let A be a τ_Gamma-closed set. Then Gamma(A) ⊆ A.

Clearly, Gamma(A) ⊆ A and A = ∅ ∈ I.

Hence, A is an R_Gamma-perfect set.

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Proposition 4.5:

If A is a subset of an ideal topological space (X, τ, I) and A ∈ I, then A is a C_Gamma-perfect set.

Proof:

Since A ∈ I, then Gamma(A) = ∅.

Clearly, A - Gamma(A) = A ∈ I

and Gamma(A) - A = ∅ ∈ I.

So, A is both L_Gamma-perfect and R_Gamma-perfect.

Example:

Let X = {a, b, c},

τ = {∅, X, {a}, {a, b}},

I = {∅, {a, b}}.

If A = {a, b}, then Gamma(A) = ∅

⇒ Gamma(A) - A = ∅

and A - Gamma(A) = ∅

⇒ (Gamma(A) - A) ∪ (A - Gamma(A)) = Delta(Gamma(A)) = ∅ ∈ I

Therefore, A is a C_Gamma-perfect set.

Corollary 4.6:

If A is a subset of an ideal topological space (X, τ, I), then the following hold:

1. If A ∈ I, then every subset of A is C_Gamma-perfect.
2. If A is R_Gamma-perfect, then Gamma(A) - A is C_Gamma-perfect.
3. If A is L_Gamma-perfect, then Gamma(A) - A is C_Gamma-perfect.
4. If A is C_Gamma-perfect, then Delta(Gamma(A)) ⊆ A is C_Gamma-perfect.

Proof:

The proof follows from Proposition 3.4.

1. Suppose A is R_Gamma-perfect, then Gamma(A) - A ∈ I. Thus we want to show that Gamma(A) is C_Gamma-perfect, which implies Gamma(A) - Gamma(Gamma(A)) = ∅ ∈ I and by proposition 3.6, Gamma - Gamma(Gamma(A)) = Gamma(A) - A ∈ I. Hence Gamma(A) is both R_Gamma-perfect and L_Gamma-perfect. Therefore, Gamma(A) - A is C_Gamma-perfect set.
2. Suppose A is L_Gamma-perfect set, then A - Gamma(A) ∈ I. Thus, we want to show that A - Gamma(A) is C_Gamma-perfect. Implies Gamma(A - Gamma(A)) = A - Gamma(A) = ∅ ∈ I implies Gamma(Gamma(A)) - A ∈ I and also A - Gamma(Gamma(A)) = A - Gamma(A) ∈ I. Hence A -

Gamma(A) is both L_Gamma-perfect and R_Gamma-perfect set and so A - Gamma(A) is C_Gamma-perfect set.

3. Suppose A is C_Gamma-perfect set, then A - Gamma(A) ∈ I and Gamma(A) - A ∈ I. By finite additive property of ideal, (A - Gamma(A)) ∪ (Gamma(A) - A) ∈ I. Hence Delta(Gamma(A)) is C_Gamma-perfect set.

Corollary 4.7:

If A is a subset of an ideal topological space (X, τ, I) and A ∩ I = ∅,

then the following hold:

1. A - Gamma(A) is C_Gamma-perfect
2. If A is R_Gamma-perfect then Gamma(A) - A is C_Gamma-perfect
3. If A is L_Gamma-perfect then A - Gamma(A) is C_Gamma-perfect

Proof:

1. The proof follows from corollary 3.10.
2. Suppose A ∩ Gamma(A) = ∅, then A ∈ I and Gamma(A) = ∅.

Thus we want to show that A - Gamma(A) = A ∈ I and also

Gamma(A) - A = ∅ ∈ I implies A is both L_Gamma-perfect and R_Gamma-perfect.

Hence A is C_Gamma-perfect set.

Proposition 4.8:

If (X, τ, I) is an ideal topological space, then every I-dense-in-itself set is L_Gamma-perfect set.

Proof:

Suppose A is I-dense-in-itself on X, then A ⊆ A.

Since by lemma 3.2, Gamma(A) ⊆ Gamma(A) implies A ⊆ Gamma(A) and Gamma(A) implies A = Gamma(A), thus A - Gamma(A) = ∅ ∈ I.

Hence A is L_Gamma-perfect set.

Corollary 4.9:

If (X, τ, I) is an ideal topological space, then the following hold:

1. Every I-dense set is L_Gamma-perfect set.

2. Every I-open set is L_Gamma-perfect set.

3. Every almost I-open set is L_Gamma-perfect set.

4. Every regular I-closed set is L_Gamma-perfect set.

Proof:

The proof follows from proposition 3.14.

Proposition 4.10

If (X, τ, I) is an ideal topological space, then ∅ and X are L_Gamma-perfect set.

Proof:

1. Since ∅ - Gamma(∅) = ∅ ∈ I. Hence ∅ is L_Gamma-perfect set.

2. If I is codense, then X = X. By lemma 3.2, X ⊆ Gamma(X), clearly X = X ⊆ Gamma(X) implies X ⊆ Gamma(X) implies X - Gamma(X) = ∅ ∈ I. Hence X is L_Gamma-perfect set.

Proposition 4.11:

Let A, B be two subsets of an ideal topological space such that A ⊆ B and Gamma(A) ⊆ Gamma(B), then the following hold:

1. B is R_Gamma-perfect if A is R_Gamma-perfect set.

2. A is L_Gamma-perfect if B is L_Gamma-perfect set.

Proof:

1. Suppose A is R_Gamma-perfect, then Gamma(A) - A ∈ I. Thus, Gamma(B) - B = Gamma(A) - B ⊆ Gamma(A) - A. Hence B is R_Gamma-perfect set.

2. Suppose B is L_Gamma-perfect, then B - Gamma(B) ∈ I.

Thus, A - Gamma(A) = A - Gamma(B) ⊆ B - Gamma(B).

Hence A is L_Gamma-perfect set.

Example:

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}} and I = {∅, {a}, {a, b}} and A = {b}, B = {a, b} then Gamma(A) = {∅}, Gamma(B) = {b} ⇒ Gamma(A) ⊆ Gamma(B), Gamma(A) - A = ∅ ∈ I and B - Gamma(B) = {a} ∈ I.

Therefore, B is R_Gamma-perfect set if A is R_Gamma-perfect set and A is L_Gamma-perfect set if B is L_Gamma-perfect set.

Corollary 4.12:

Let A, B be two subsets of an ideal topological space such that A ⊆ B ⊆ Cl(A), then the following hold:

1. B is R_Gamma-perfect if A is R_Gamma-perfect set.

2. A is L_Gamma-perfect if B is L_Gamma-perfect set.

Proof: The proof follows from proposition 3.20

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RELATIONSHIP BETWEEN L_*–PERFECT, R_*–PERFECT AND L_GAMMA–PERFECT, R_GAMMA–PERFECT

In this section we investigste the relationship between the exiting $L_{Æ}$-perfect, RÆ-perfect and L -perfect, R -perfect,

Proposition 5.1:

Every L_*–perfect is L_Gamma–perfect but the converse is not true.

Proof:

Suppose A is L_*–perfect, then A - A ∈ I.

By lemma 3.2, A ⊆ Gamma(A) implies A - Gamma(A) ⊆ A - A ∈ I.

Hence A is L_Gamma–perfect set.

Proposition 5.2:

Every R_Gamma–perfect is R_*–perfect set but the converse is not true.

Proof:

Suppose A is R_Gamma–perfect, then Gamma(A) - A ∈ I.

Since A ⊆ Gamma(A), clearly A - A ⊆ Gamma(A) - A ∈ I.

Hence A is R_*–perfect set.

Proposition 5.3:

If A, B are R_Gamma–perfect, then A ∪ B is R_Gamma–perfect set.

Proof:

Since A and B are R_Gamma–perfect, then Gamma(A) - A ∈ I and Gamma(B) - B ∈ I.

Then Gamma(A ∪ B) - (A ∪ B) ⊆ (Gamma(A) - A) ∪ (Gamma(B) - B) ∈ I.

Hence A ∪ B is R_Gamma–perfect set.

Proposition 5.4:

If A and B are L_Gamma–perfect, then A ∪ B is L_Gamma–perfect set.

Proof: Suppose A and B are L_Gamma–perfect, then A - Gamma(A) ∈ I and B –

Gamma(B) ∈ I.

Then (A ∪ B) - Gamma(A ∪ B) ⊆ (A - Gamma(A)) ∪ (B - Gamma(B)) ∈ I.

Hence A ∪ B is L_Gamma–perfect set.

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Corollary 5.5:

In an ideal topological space, the following hold:

1. Finite union of R_Gamma–perfect is R_Gamma–perfect.
2. Finite union of L_Gamma–perfect is L_Gamma–perfect.

Proposition 5.6:

If A and B are R_Gamma–perfect, then A ∩ B is R_Gamma–perfect set.

Proof: Suppose A and B are R_Gamma–perfect, then Gamma(A) - A ∈ I and Gamma(B) - B ∈ I.

Then Gamma(A ∩ B) - (A ∩ B) ⊆ (Gamma(A) - A) ∪ (Gamma(B) - B) ∈ I.

Hence A ∩ B is R_Gamma–perfect set.

Proposition 5.7:

If A and B are L_Gamma–perfect, then A ∩ B is L_Gamma–perfect set.

Proof: Suppose A and B are L_Gamma–perfect, then A - Gamma(A) ∈ I and B - Gamma(B) ∈ I.

Then (A ∩ B) - Gamma(A ∩ B) ⊆ (A - Gamma(A)) ∪ (B - Gamma(B)) ∈ I.

Hence A ∩ B is L_Gamma–perfect set.

Corollary 5.8:

In an ideal topological space (X, τ, I), the following hold:

1. Finite intersection of R_Gamma–perfect is R_Gamma–perfect.

2. Finite intersection of L_Gamma–perfect is L_Gamma–perfect.

Proof: The proof follows from propositions above.

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