Numerical Analysis of Differential Equations Using Haar Wavelets

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ABSTRACT

Wavelet analysis is a rapidly developing field of mathematical and application-oriented research across many scientific disciplines, receiving increasing attention in engineering initiatives. The wavelet transform is localized in both space (time) and frequency, allowing it to extract information from signals that are often impossible to analyze using Fourier or even windowed Fourier transforms. Wavelets serve as effective tools, providing robust mathematical models for scientific phenomena typically represented through linear or nonlinear differential equations. In the present paper, we discuss the Haar wavelet operations and examine the variation of the translation parameter (k) and the dilatation parameter (j); we also consider the second-order differential equations of motion to assess the efficiency and applicability of the intended method. Interestingly, the analysis indicates that the error decreases exponentially as the resolution level increases, leading to more accurate results.

Keywords: Haar wavelet; Collocation method; simple harmonic motion;

INTRODUCTION

Various categories of engineering and real-world issues are described and analyzed as ordinary differential equations (ODEs) [1]. Although certain problems can be addressed through analytical techniques, not all scenarios lend themselves to analytical solutions. Nevertheless, the absence of a suitable analytical approach is not the only reason for opting for numerical methods. In practice, even when an analytical solution is achievable, the computational resources required may be prohibitively high [2]. Knowing only the differential equation is insufficient for finding the solutions to the problem, suggesting that additional information is necessary. Since the proposed solution to the given ODE is an initial value problem (IVP), extra information can be described as "Initial Conditions. " In recent years, the study of IVPs in second-order ordinary differential equations (ODEs) has attracted the attention of many mathematicians and physicists [3, 4]. For the numerical approximation of ODEs, these properties have provided a solid foundation for the use of Haar functions[5].

Various other numerical methods have previously been employed to tackle these challenges [6]. In this research, the Haar Wavelet Method is utilized to numerically solve these differential equations. This approach was first introduced by Chen and Hsiao in 1997 and subsequently applied to the Haar wavelet collocation method (HWCM) to address differential and integral equations. In 2008, Phang Chang and Phang Piau [7] developed operational matrices for Haar Wavelets to solve ODEs, conducting all calculations with the matrix representation of wavelets and their integrals, thereby simplifying the process. Recently, Mittal and Pandit created and utilized novel scale- 3 Haar wavelets for fractional dynamical systems [8] and engineering applications [9], demonstrating their benefits over conventional Haar wavelets.

The goal of this research is to evaluate the quality, efficiency, and precision of the Haar Wavelet Method for solving ODEs with given initial conditions. The precision and detail of the technique can be easily adjusted by modifying the value of J, which will be discussed in the upcoming sections. Typical computation time for the solution of such problems at J = 6 yields very low errors and takes just a few seconds on an average commercial computer with 2-core processors. Considering the necessary computational power and minimal error deviation, the suggested approach demonstrates the efficacy of the Haar Wavelet Method.

Haar wavelet and its integrals

In the present paper, we obtain an orthogonal basis for the subspaces of L^2 [a, b] called the Haar wavelet family. For this notations introduced by Lepik (2014) [8, 10] were used. The interval [a, b] is divided into 2^{J+1} subintervals of equal length $\left(\Delta t = \frac{(b-a)}{2^{J+1}}\right)$, where J is called the maximal level of resolution. The two parameters such as the dilation parameter values j = 0, 1, 2, ..., J and the translation parameter k = 0, 1, 2, ..., $2^j - 1$. With these two parameters, the i^{th} Haar wavelet in the Haar family is defined as

$$h_i(t) = \begin{cases} 1, & \text{for } t \in [\xi_1(i), \xi_2(i)), \\ -1, & \text{for } t \in [\xi_2(i), \xi_3(i)), \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

here i = m + k + 1 , $\xi_1(i) = a + 2k\mu\Delta t$, $\xi_2(i) = a + (2k+1)\mu\Delta t$, $\xi_3(i) = a + 2(k+1)\mu\Delta t$, where $\mu = 2^{J-j}$.

Above equations are valid for i > 2 . $h_1(t)$ and $h_2(t)$ are called father and mother wavelets in Haar wavelet family and are given by

$$h_1(t) = \begin{cases} 1, & \text{for } t \in [a,b), \\ 0, & \text{otherwise,} \end{cases}$$

$$h_2(t) = \begin{cases} 1, & \text{for } t \in \left[a, \frac{a+b}{2}\right), \\ -1, & \text{for } t \in \left[\frac{a+b}{2}, b\right), \\ 0, & \text{otherwise,} \end{cases}$$

Any function which is having finite energy on [a,b], i.e. $f \in L^2[a,b]$ can be decomposed as an infinite sum of Haar wavelets:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \quad (2)$$

where a_i's are called Haar coefficients. If f is either piecewise constant or wish to be approximated by piecewise constant on each subinterval, then the above infinite series will be terminated at a finite number of terms.

Since we have an explicit expression for each member of the Haar family [11]. We can integrate as many times depend on the application. The following notations are used for γ times of integration of members in the family defined on [a,b):

$$P_{\gamma,i}(t) = \int_a^t \int_a^t \ldots \int_a^t h_i(x)\, dx^\gamma ,$$

$$E_{\gamma,i} = \int_a^b P_{\gamma,i}(t)\, dt$$

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For i=1 , (8) becomes

$$P_{\gamma,i}(t) = \frac{1}{\gamma!}(t-a)^{\gamma},$$

For $i \geq 2$, we have

$$P_{\gamma,i}(t) = \begin{cases} 0, if\ t\ \in [a,\xi_1(i)), \\ \frac{1}{\gamma!}(t-\xi_1(i))^{\gamma}, if\ t\ \in [\xi_1(i),\xi_2(i)), \\ \frac{1}{\gamma!}\{(t-\xi_1(i))^{\gamma} - 2(t-\xi_2(i))^{\gamma}\}, if\ t\ \in [\xi_2(i),\xi_3(i)), \\ \frac{1}{\gamma!}\{(t-\xi_1(i))^{\gamma} - 2(t-\xi_2(i))^{\gamma} + (t-\xi_3(i))^{\gamma}\}, if\ t\ \in [\xi_3(i),b). \end{cases}$$

For $J = 2$, The Haar matrix H and the integrated Haar matrices p_1 are as follows

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.4375 & 0.3125 & 0.1875 & 0.0625 \\ 0.0625 & 0.1875 & 0.1875 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.1875 & 0.1875 & 0.0625 \\ 0.0625 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0625 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.0625 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0625 & 0.0625 \end{bmatrix}$$

The infinite series of Haar wavelets for the function that possesses finite energy over the interval$[a, b]$, i.e. $f \in L^2[a, b]$ is

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{3}$$

Here a_i*'s* are referred to as the coefficients of the Haar wavelet.

If f is a piecewise constant on each subinterval in which the series terminates to a finite.

Numerical Studies

We considered the examples whose exact solutions are known arising in simple harmonic motion. Effectiveness of Haar wavelet collocation method (HWCM) was investigated problem and compared with the exact solution represented the same in the form of graph and Table as below.

RESULTS AND DISCUSSION

In this paper, we have considered the simple harmonic second-order equation is

$$\frac{d^2x}{dt^2} + 2x = 0 \qquad (4)$$

$0 \le t \le 6$with initial conditions $x(0) = 0 \& x'(0) = 1,$ and the exact solution is $x = \frac{1}{2} sin\, 2\, t$

Figure 1. Indicates exact solution and Haar Wavelet solution with maximum resolution number J is designated as 6 for this solution. It's clearly observed that an Eq. (4) Haar wavelet collocation method coincides with the exact solution and the obtained results are well matched with the periodic motion. The numerical solutions obtained using the Haar wavelet collocation approach and the exact solutions with absolute error are compared in Table 1 and it indicates the minimum error established between Haar wavelet collocation method and the exact solution with maximum resolution J=6. On the other hand, more collocation points lead to more accurate and error-free outcomes. Furthermore, these findings offer the appropriate level of precision for real-world problems.

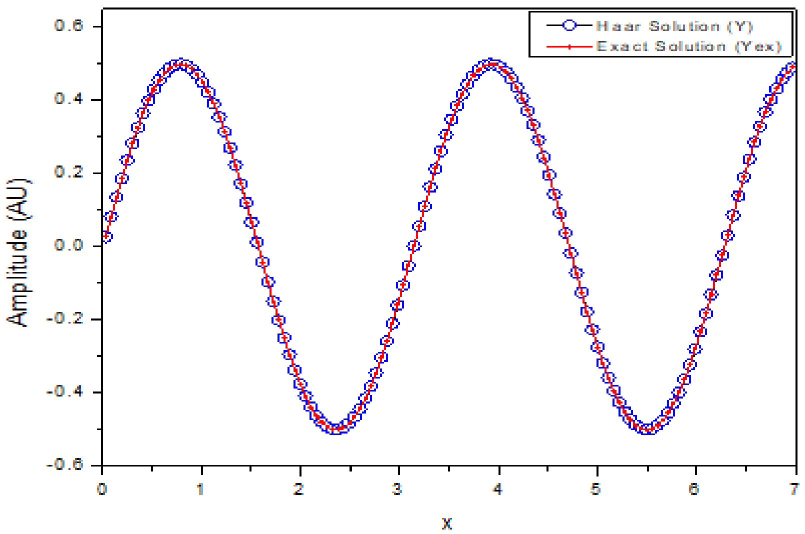

Figure 1. Indicates exact solution and Haar Wavelet solution with maximum resolution number J is designated as 6 for this solution.

Table 1 represents the numerical results, exact solution and absolute errors for example 1 with J=6

t	Haar solution	Exact solution	Absolute error
0.0273	0.0273	0.0273	0.0000
0.5195	0.4306	0.4310	0.0004
1.0117	0.4494	0.4496	0.0002
1.5039	0.0670	0.0667	0.0003
2.0508	-0.4090	-0.4096	0.0006
2.5430	-0.4654	-0.4655	0.0001
3.0352	-0.1063	-0.1056	0.0007
3.5273	0.3477	0.3486	0.0009
4.0195	0.4913	0.4915	0.0002
4.5117	0.1962	0.1953	0.0009
5.00039	-0.2740	-0.2753	0.0012
5.508	-0.4970	-0.4972	0.0002
6.0430	-0.2322	-0.2311	0.0012

CONCLUSION

The Haar wavelet is utilized in various fields, including both linear and nonlinear ordinary differential equations (ODEs), as mentioned earlier in this study. The research highlights the effectiveness of the Haar Wavelet, which is part of the Wavelet family, in addressing different types of initial value problems for ODEs. The outcomes calculated using the Haar wavelet method are compared to their exact solutions. As a result, the numerical values and graphs demonstrate that this method is effective and allows for quick results through the use of Haar matrices.

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