Comparison Theorems for Weak Topologies (2)

[1]Chika S. Moore, [1]Alexander O. Ilo, [2]Ifeanyi Omezi

[1]Department of Mathematics, Nnamdi Azikiwe University, P.M.B. 5025, Awka, Anambra State

[2]Department of Petroleum Engineering, Nnamdi Azikiwe University, P.M.B. 5025, Awka, Anambra State

DOI: https://doi.org/10.51584/IJRIAS.2025.1001018

Received: 06 August 2024; Accepted: 28 August 2024; Published: 06 February 2025

ABSTRACT

Weak topology on a nonempty set X is defined as the smallest or weakest topology on X with respect to which a given (fixed) family of functions on X is continuous. Let τ_w be a weak topology generated on a nonempty set X by a family

$\{f_\alpha: \alpha \in \Delta\}$ of functions, together with a corresponding family

$\{(X_\alpha, \tau_\alpha): \alpha \in \Delta\}$ of topological spaces. If for some $\alpha_0 \in \Delta$, τ_{α_0} on X_{α_0} is not the indiscrete topology and f_{α_0} meets certain requirements, then there exists another topology τ_{w_1} on X such that τ_{w_1} is strictly weaker than τ_w and f_α is τ_{w1}-continuous, for all $\alpha \in \Delta$. Here in Part 2 of our Comparison Theorems for Weak Topologies,

1. We showed that not every weak topology τ_w has a strictly weaker weak topology τ_{w1}.

2. We constructed important examples to show (a) that a weak topological system may not have a strictly weaker weak topology, (b) that a weak topological system can have a strictly weaker weak topology, and (c) that a weak topological system can have both comparable and non-comparable weak topologies.

3. A further research agenda is (now) set to find out when and why we must use a particular weak topology (instead of the others) in any given context of analysis.

Key Words: Topology, Weak Topology, Weak Topological System, Product Topological System, Chain of Topologies, Strictly Weaker Weak Topologies, Pairwise Strictly Comparable Weak Topologies

Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

MAIN RESULTS—EXAMPLES AND MORE GENERAL PROOFS

CASE I—τ_{w1} Does Not Exist for Every Weak Topology τ_w

That is, not every weak topology has a strictly weaker weak topology.

EXAMPLE 1:

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system in which each of the topological range spaces is an indiscrete space and each of the functions is onto. Then necessarily (X, τ_w) is an indiscrete weak topological space; hence $\tau_w = \{X, \emptyset\}$[1] has no strictly weaker weak topology τ_{w1}.

EXAMPLE 2:

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system in which each of the topological range spaces is

[1] This does not mean that the cardinality of such a weak topology is, in general, 2.

an indiscrete space. Let the domain of one of the functions (say f_β) not be all of X, say the domain of f_β is a proper subset A of X. Then necessarily (X, τ_w) is an indiscrete weak topological space; and $\tau_w = \{X, \emptyset, A\}$ has no strictly weaker weak topology τ_{w_1}.

It is worth pointing out that if the only thing different between examples 1 and 2 above is the function f_β, then the two weak topologies are comparable (as can easily be seen), but the two weak topological systems are, strictly speaking, totally different. Because of this fact, the two weak topologies are not comparable weak topologies, as the families of functions that generate them are different. However, in examples 3 and 4 below the families of functions are the same; hence the weak topologies in the two examples are comparable.

CASE II—τ_{w1} Exists for Many Weak Topologies τ_w

Many weak topologies have strictly weaker weak topologies. In Corollary 2.1 of ***Comparison Theorems for Weak Topologies (1)***, it is proved that the usual weak and weak star topologies have chains of pairwise strictly comparable weaker weak or weak star topologies. (See [3]) Here we are set to fulfill the objective 2 of our abstract.

EXAMPLE 3:

Let $X = \{0,1\}$. The Sierpinski topology on X is the collection $\tau = \{\emptyset, X, \{0\}\}$. The Cartesian product of X with itself is the set $\bar{X} = X \times X = \{(0,0), (0,1), (1,1),(1,0)\}$ of 4 coordinate points. We can define the projection maps p_i: $\bar{X} \to X$; for $i = 1,2$ in the usual way by $p_i\{(x, y)\} = x$ if $i = 1$, and $p_i\{(x,y)\} = y$ if $i = 2$. Let us also endow each factor space X_1 and X_2 of $\bar{X}$ with this Sierpinski topology. Then we have obtained all the conditions for a product topological system $[(X, \tau_w),\{(X_\alpha,\tau_\alpha)\},\{p_\alpha\}]$ where the family of functions is made up of only two projection maps; and the product topology is the family

$$\tau_w = \{\emptyset, X, \{(0,0),(0,1)\},\{(0,0),(1,0)\},\{(0,0)\},\{(0,0),(1,0),(0,1)\}\}$$

of 6 subsets of $\bar{X}$.

EXAMPLE 4:

Now let us endow only one factor space of X with the Sierpinski topology, and the remaining factor space with the indiscrete topology. The product (weak) topology that would now emerge on $\bar{X}$ is seen to be

$$\tau_{w1} = \{\emptyset, \mathrm{X}, \{(0,0),(1,0)\}\},$$

a family of only 3 subsets of $\bar{X}$. It is also easily seen that τ_{w1} is a strictly weaker weak topology than τ_w, on $\bar{X}$. Yet both weak topologies are generated by the same fixed family of functions. One interesting question now is: Which of the two weak topologies of examples 3 and 4 (generated by the same family of functions) should be considered *the* weak topology generated by these functions, and why should we prefer to use one of them instead of the other in a context of analysis of this family of functions?

EXAMPLE 5:

The Euclidean (or usual) topology of the Cartesian plane R^2 is known as the weak topology τ_w of the plane when its factor spaces R_1, R_2 (respectively the horizontal and the vertical axes) are themselves given their usual (Euclidean) topology, and the projection maps are the family of functions.

If we endow any of the axes of the plane R^2 with a topology strictly weaker than the usual topology of R the weak topology that would then be generated on the plane by the projection maps would be strictly weaker than (what may now be called) the usual weak topology of the plane. And only a second thought is all we need to see that virtually every topology on an axis of the Cartesian plane R^2 has a strictly weaker topology—hence virtually every weak topology (including of course product topology) on the plane has a strictly weaker weak (or product) topology. This somewhat strong statement will find illustration in further examples and propositions here.

EXAMPLE 6:

Let $X = (a, b) \in U$ be a fixed open interval in the usual topology U of R. Let $\gamma = \{G \in U: G \subset X\}$. Then it is easy to see that γ is a topology on X. If we now let $\tau = \gamma \cup \{R\}$, we see that τ is a topology strictly weaker than U on R. If we have the two factor spaces of R^2 endowed with the topology τ and have the projection maps as the family of functions on R^2, the weak (product) topology now on the plane R^2 would be strictly weaker than the usual weak topology of the plane.

EXAMPLE 7:

Let $n \in N$ be a natural number, and let $X_n = (-n, n) \in U$, a U-open interval, where U is the usual topology on R. We can let τ_n be the topology induced on R by its U-open subset X_n following the process of construction in example 3 above. Then we observe the following.

1. Each τ_n on R is strictly weaker than the usual topology U on R for all $n \in N$. Hence by endowing each factor space of R^2 with τ_n we can obtain a strictly weaker weak topology (than the Euclidean topology) on R^2, generated by the projection maps.

2. If $m > n$ then τ_n is strictly weaker than τ_m on R. Hence corresponding to any pair m, n of natural numbers there exists a pair τ_m and τ_n of strictly comparable and strictly weaker topologies than U on R.

3. Hence corresponding to any pair m, n of natural numbers there exists a pair τ_{w_m} and τ_{w_n} of strictly comparable and strictly *weaker* weak topologies than the usual weak topology τ_w on R^2. Hence

4. There exists a chain $\{\tau_{w_n}\}_{n\in N}$ of pairwise strictly comparable and strictly weaker weak topologies than the usual weak topology τ_w on R^2 in that

$$\tau_{w_1} < \tau_{w_2} < \tau_{w_3} < \cdots < \tau_w$$

5. As $n \to \infty$, $\tau_{w_n}\} \to \tau_w$.

6. And finally, any nonempty subset of the set R of real numbers can be used as the indexing set here in place of N and the subset-induced topologies can be constructed in many other ways than what is done here.

Remark:

1. The analysis above, particularly in example 4, copiously holds for any weak topology on any nonempty set which has a range (topological) space that in turn has a strictly weaker topology. And this scenario is a very fortuitous one as it tells us that we can seek and find a strictly weaker weak topology τ_{w_1}, than τ_w, provided τ_w is not an indiscrete weak topology; that we can further seek and find a strictly weaker weak topology τ_{w_2}, than τ_{w_1}, provided τ_{w_1} is not an indiscrete weak topology; and so on.

2. All the range (topological) spaces must not be endowed with only one type of topology in order to get a strictly weaker weak topology than a given weak topology.

3. The expositions in the examples above can be extended to (particularly) general Euclidean topology of R^n—and in general, to many weak topological systems.

4. From the observations above it is clear that every pair of strictly comparable topologies in a range space of a weak topological system equally has correspondingly a pair of strictly comparable weaker weak topologies generated (if it can be so said) by them. This is a very important result which we state below in lemma 1.1. (In the following lemma, it is assumed that the function f_r in the weak topological system meets the conditions of lemma 2.1 of ***Comparison Theorems for Weak Topologies (1)***.)

Lemma 1.1 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ *be a weak topological system. If in a range space, say* (X_r, τ_r) *there exist two strictly comparable topologies* τ_{r1} *and* τ_{r2} *where, say* $\tau_{r_1} < \tau_{r_2}$ *(and both are strictly weaker than* τ_r*), then there exist two strictly comparable weaker weak topologies* τ_{w_1} *and* τ_{w_2} *on X, in that* $\tau_{w_1} < \tau_{w_2} < \tau_w$

EXAMPLE 5:

It is known that a finite product of discrete topological spaces is discrete. We add that if the cardinality of any of the factor spaces of a finite dimensional discrete product space is greater than 1, then such a discrete product topology has a strictly weaker product topology.

The strictly weaker weak topologies obtained in respect of a given weak topology may not be pairwise strictly comparable; in fact they may not be comparable at all. The next example illustrates this. That is, if we look at the foregoing examples it may appear that all the strictly weaker weak topologies τ_{w_i} (when they exist) of a weak topology τ_w are always pairwise comparable. This is not actually so.

Definition 1.1 *Let*

$$K = \{\tfrac{1}{n}\}_{n \in N} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots\} \subset [0,1],$$

and let (R, U) *denote R with its usual topology U. Let* $B = (a, b) - K$*, where* (a, b) *is an open interval of the set of real numbers with its usual topology. (We observe that* $(a, b) - K = (a, b)$*, if* $(a,b) \cap K = \emptyset$*.) Then the K-topology on the set R of real numbers is the collection* $\tau_K = U \cup \{(a_\alpha, b_\alpha) - K\}_{\alpha \in \Delta}$*; the union of sets of type B together with U.*

EXAMPLE 6:

We observe that the topology τ_{K_1} on the set R of real numbers, given by the collection $\tau_{K_1} = \{R\} \cup \{(a_\alpha, b_\alpha) - K\}_{\alpha \in \Delta}$ of sets of type B together with R itself, is strictly weaker than the K-topology on R. Also, the K_1-topology and the usual topology on R are not comparable. To see this, we observe that (for instance) the U-open interval (0,1) is not K_1-open; and that the K_1-open set

$$G = (-1, 1) - K = (-1, 0] \bigcup \left(\bigcup_{n=1}^{\infty} (\tfrac{1}{n+1}, \tfrac{1}{n}) \right) \quad . \; . \; . \; (*)$$

is not U-open.

Construction 1 (K-topology-induced weak topology) *Suppose the Cartesian plane* R^2 *has the projection maps defined on it, as usual, and that the factor spaces* R_1 *and* R_2 *(respectively horizontal and vertical) are each endowed with the K-topology. The K-topology-induced weak topology of* R^2 *is the weak topology generated on* R^2 *by the projection maps under this arrangement; i.e. where the factor spaces are given the K-topology.*

We may want to know one or two things about the landscape of this topology. Suppose $G = [(a, b) - K] \in K_R$ is an arbitrary open set in the K-topology of R. Then two cases arise: namely, that either $(a, b) \cap K = \emptyset$ or $(a, b) \cap K \neq \emptyset$.

Suppose, first, that $(a, b) \cap K \neq \emptyset$. Then $p^{-1}_1(G) = \{\bar{x} \in R^2 : p_1(\bar{x}) \in G\} = \{\bar{x} \in R^2 : p_1(\bar{x}) \in [(a, b) - K]\} = \{(x_1, x_2) \in R^2 : x_1 \in [(a, b) - K]\} = \{(x_1, x_2) \in R^2 : x_1 \in (a, b)$ and $x_1 \notin K\}$. This is an open vertical infinite strip with deleted infinite vertical lines through the common points of (a, b) and K.

If $(a, b) \cap K = \emptyset$, then $p_1^{-1}(G) = \{\bar{x} \in R^2 : p_1(\bar{x}) \in (a, b)\} = \{(x_1, x_2) \in R^2 : x_1 \in (a, b)\}$. This is the usual open vertical infinite strip, with no demarcations in it.

In the same way, $p_2^{-1}(G)$ will either be an open horizontal infinite strip with deleted horizontal infinite lines or

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue I January 2025

the usual horizontal infinite strips, without demarcations.

If, however, we replace K-topology by the K_1-topology in the above construction— thereby having instead the K_1-topology induced weak topology of R^2—then by sketching the geometrical picture of the sub-basic and basic sets of this topology on R^2 it will be found that the usual open rectangles, but not all, are open in this K_1-topology induced weak topology. Reason: Let $(a,b) = (0, {}^3/_2)$. Then the set $p_1{}^{-1}(a,b) \cap p_2{}^{-1}(a,b)$ would be an ordinary (or usual) plane rectangle in R^2, hence open in the usual topology of R^2, generated by the projection maps when the factor spaces of R^2 are endowed with their own usual topologies of R. But this particular rectangle is not open in the K_1-topology-induced weak topology of R^2 since it cannot be expressed as the union of any number of open sets in the K_1-topology-induced topology. Conversely the set B, as a set in the usual topology of R^2 is not contained in any open set in the K_1-topologyinduced topology; hence it is not open in this K_1-topology-induced topology; though it is open in the K-topology induced weak topology. The reason for this is that the set B turns out to be what may be called a *squared* rectangle in the K_1-topology-induced topology.

Also this K_1-topology-induced weak topology is not weaker than the usual Euclidean topology of the plane since the set $(*)$ above is not $U - open$. In summary of this example therefore, if we let $\tau_{w_K}, \tau_{w_{K_1}}$ and τ_w denote respectively the K-topology-induced weak topology, the K_1-topology-induced weak topology and the usual weak topology of the Cartesian plane R^2, then we see that

- $\tau_w < \tau_{w_K}$;
- $\tau_{w_{K_1}} < \tau_{w_K}$; and that
- $\tau_{w_{K_1}}$ and τ_w are not comparable.

NOTE:

1. Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If, for some $r \in \Delta$, τ_r has two distinct strictly weaker topologies τ_{r_1} and τ_{r_2} then it is clear from the foregoing that we can get a strictly weaker weak topology τ_{w_1}, than τ_w, on X in at least two ways.
2. The only weak topology which we know (for now) has no strictly weaker weak topologies is the indiscrete weak topology (whose range spaces are all indiscrete topological spaces). This implies that any non-indiscrete weak topology has a strictly weaker weak topology. The last assertion is clearly an important statement which needs to be proved. The proof of this will be given below in theorem 1.1.
3. In terms of topological properties (like the separation axioms, compactness, etc.) there is now a challenge to identify or characterize the weak topologies whose strictly weaker weak topologies inherit their property; and it will equally be important and interesting to find those topological properties that are preserved under the operation of getting strictly weaker weak topologies.

Lemma 1.2 *Let τ and η be two topologies on a set X and let S_τ and S_η denote the subbases for τ and η respectively. Then $S_\tau \subset S_\eta \Rightarrow \tau \leq \eta$.*

Proof:

Let

$$B_\tau = \left\{ \bigcap_{i=1}^{n} G_i : G_i \in S_\tau \right\}$$

be the base for τ and let

$$B_\eta = \left\{ \bigcap_{i=1}^{n} G_i : G_i \in S_\eta \right\}$$

be the base for η. If $S_\tau \subset S_\eta$ then clearly $B_\tau \subset B_\eta$, and hence that τ =

$$\left\{ \bigcup_{\alpha\in\Delta} B_\alpha : B_\alpha \in B_\tau \right\}$$
$$\tau \leq \eta.$$

is a subfamily of

$$\eta \ = \ \left\{ \bigcup_{\alpha\in\Delta} B_\alpha : B_\alpha \in B_\eta \right\}.$$

That is, $\tau \leq \eta$.

■

Clearly the following result has many fundamental and far-reaching implications.

Theorem 1.1 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* $\exists$ (X_r, τ_r), *for some* $r \in \Delta$, $\ni$ $Card(\tau_r) > 2$ *then* τ_w *has a strictly weaker weak topology.*

Proof:

$Card(\tau_r) > 2$ implies that τ_r contains at least 3 subsets of X_r. So, let $\tau_r = \{\emptyset, X_r, G\}$, where G is a nonempty proper subset of X_r. Then $\tau_{r1} = \tau_r - \{G\}$ is a topology on X_r strictly weaker than τ_r. Let τ_{w_1} be the weak topology generated on X by the fixed family of functions when X_r has the topology τ_{r_1} and the remaining range spaces have their topologies unchanged. Then τ_{w_1} is strictly weaker than τ_w since in particular $f_r^{-1}(G) \notin \tau_{w_1}$. The proof is complete.

■

The meaning of theorem 1.1 is that a weak topology τ_w generated on a set X by a given family F of functions has a strictly weaker weak topology τ_{w_1} on X generated by the same family F of functions provided one of its range spaces is not an indiscrete topological space.

The theorem again has this very important implication which we state below as a corollary.

Corollary 1.1 *Every non-indiscrete weak topology on a nonempty set X is at the peak of a chain of pairwise strictly comparable weaker weak topologies.*

Note

The cardinality of such a chain will depend on (a) the cardinality of X and (b) the creative way we choose to develop the chain. If X is a finite set, then the chain will necessarily be finite; and if X is infinite the chain can be made to be finite or infinite. The usual Euclidean topologies of R^n $(n \geq 2)$, as weak topologies, can have finite chain, denumerable chain, or uncountable chain of pairwise strictly comparable *weaker* weak topologies.

Proposition 1.1 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If X is finite and* τ_w *is not indiscrete, then* τ_w *has a strictly weaker weak topology which makes X compact.*

Though proposition 1.1 is a simple result, its generalization or extension (neither of which is available now) will not be a negligible achievement since 'compactness' is a very important issue in the whole of general topology.

Proposition 1.2 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. Let* τ_{w_i} *denote the weak topology on X when* X_α *has the topology* τ_{α_i}.

Then

1. $\tau_{\alpha_i} \leq \tau_{\alpha_j} \Rightarrow \tau_{w_i} \leq \tau_{w_j}$
2. τ_{α_i} and τ_{α_j} *not comparable, implies* τ_{w_i} and τ_{w_j} *not comparable;*
3. $\tau_{\alpha_i} < \tau_{\alpha_j}$ *and* $\tau_{r_i} > \tau_{r_j}$ *implies* τ_{w_i} and τ_{w_j} *not comparable; and*
4. $\{\tau_{\alpha_i}\}$, *a chain, implies that* $\{\tau_{w_i}\}$ *is a chain.*

Proof:

1. Lemma 1.2 makes this easy to see.
2. If τ_{α_i} and τ_{α_j} are not comparable, then the subbases of τ_{w_i} and τ_{w_j} (and hence the topologies τ_{w_i} and τ_{w_j}) are not comparable.
3. If $\tau_{\alpha_i} < \tau_{\alpha_j}$ then from the foregoing, $\tau_{w_i} < \tau_{w_j}$. But then $\tau_{ri} > \tau_{rj}$ implies that $\tau_{w_i} > \tau_{w_j}$. That is, $\tau_{w_i} < \tau_{w_j}$ and $\tau_{w_i} > \tau_{w_j}$. This is a contradiction; implying that τ_{w_i} and τ_{w_j} are not comparable.
4. $C = \{\tau_{\alpha_i}\}$ being a chain implies that the topologies in C are pairwise comparable. Lemma 3.1 then implies that the family $\{\tau_{w_i}\}$ of weak topologies on X is also in chain.

EXAMPLE 1

Let $X = \{a, b, c\}$, $X_1 = \{x, y\}$ and $X_2 = \{p, q, r, s, t\}$. Let f_1: $X \rightarrow X_1$ be a function defined by $f_1(a) = x$, $f_1(b) = x$ and $f_1(c) = y$. Let f_2: X $\rightarrow X_2$ be a function defined by $f_2(b) = q$ and $f_2(c) = p$. Let $\tau_1 = \{X_1, \emptyset\}$ be the topology on X_1 and let $\tau_2 = \{X_2, \emptyset\}$ be the topology on X_2. Then (X_1, τ_1) and (X_2, τ_2) are indiscrete topological spaces and the cardinality of each of the range topologies is 2. It can easily be verified that the weak topology τ_w on X generated by the family $F = \{f_1, f_2\}$ of these two functions is $\tau_w = \{\emptyset, X, \{b, c\}\}$; a family of 3 subsets of X.

EXAMPLE 2

Let $X = \{a, b, c\}$, $X_1 = \{x, y\}$ and $X_2 = \{p, q, r, s, t\}$. Let f_1: $X \rightarrow X_1$ be a function defined by $f_1(a) = x$ and $f_1(b) = x$. Let f_2: $X \rightarrow X_2$ be a function defined by $f_2(b) = q$ and $f_2(c) = p$. Let $\tau_1 = \{X_1, \emptyset\}$ be the topology on X_1 and let $\tau_2 = \{X_2, \emptyset\}$ be the topology on X_2. Then (X_1, τ_1) and (X_2, τ_2) are indiscrete topological spaces and the cardinality of each of the range topologies is 2. Now the weak topology τ_w on X generated by the family $G = \{f_1, f_2\}$ of two functions is $\tau_w = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}\}$; a family of 5 subsets of X. It is important to observe that the family F of functions in example 1 is different from the family G of functions in example 2. This observation will help us not to think that a fixed family of functions can generate two indiscrete weak topologies on the same set—as really a fixed family of functions cannot generate more than one indiscrete weak topology on a set. And the indiscrete weak topology of a family of functions must emerge only when all the range topologies are themselves indiscrete.

An indiscrete weak topology may also emerge in the usual form (with cardinality 2) in which we have known indiscrete topologies.

EXAMPLE 3

Let X, X_1 and X_2 all be as given in example 2 above and let X_1 and X_2 retain their indiscrete topologies. If the domain of f_1 is all of X and the domain of f_2 is all of X, then the weak topology τ_w on X generated by these two functions will be $\tau_w = \{\emptyset, X\}$; with cardinality 2. So, when we say *an indiscrete weak topology* we only know or mean that it is one which has no strictly weaker weak topology; the matter of the determination of its

cardinality is something else.

Proposition 1.3 *An indiscrete weak topology can have cardinality greater than 2; however, it cannot have a strictly weaker weak topology in its own system.*

Since we have seen (from examples 1 and 2 above) that an indiscrete weak topology can have cardinality greater than 2, since such an indiscrete weak topology is also a topology in the ordinary sense and hence can further be reduced in some sense (though not as a weak topology), we have yet another very important exposition.

Theorem 1.2 *Let* $[(X, \tau_w), \{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. The following statements are equivalent.*

(a) The weak topology τ_w *is not reducible to a strictly weaker weak topology in any sense.*

(b) All the range topologies of τ_w*, including any which may itself be a weak topology, have cardinality 2.*

(c) τ_w *is an indiscrete weak topology.*

Proof:

(a) If the weak topology τ_w is not reducible as a weak topology in any sense, then all the range topologies have cardinality 2; for if a range topology has cardinality greater than 2, theorem 3.5 would imply that τ_w has a strictly weaker weak topology. That is, (a) implies (b).

(b) Clearly τ_w is an indiscrete weak topology if all the range topologies of τ_w have cardinality 2.

(c) implies (a) by definition.

SUMMARY AND CONCLUSION

1. If all the range spaces are indiscrete topological spaces in the usual sense of having topologies of cardinality 2, it does not follow or mean that the *weak topology*—being then an indiscrete weak topology—would have cardinality equal to 2.

2. If the topology of a range space of a weak topological system has cardinality greater than 2, then the weak topology has a strictly weaker weak topology.

3. If there is a chain of pairwise comparable topologies in a range space of a weak topological system, then the weak topology of the system has correspondingly a chain of pairwise comparable weak topologies.

4. An indiscrete weak topology may or may not have cardinality greater than 2.

5. Clear examples are given at each stage to illustrate and demonstrate the developments/achievements being made.

REFERENCES

1. Angus E. Taylor and David C. Lay; Introduction to Functional Analysis; Second Edition, John Wiley and Sons, New York, 1980.
2. Benjamin T. Sims; Fundamentals of Topology; Macmillan Publishing Co., Inc., New York; Collier Macmillan Publishers, London and Canada, 1976.
3. Chika S. Moore, Alexander O. Ilo and Ifeanyi Omezi; Comparison Theorems for Weak Topologies (1); International Journal of Research and Innovation in Applied Science (IJRIAS); Volume IX, Issue VIII, August, 2024.
4. Edwards R.E.; Functional Analysis: Theory and Applications; Dover Publications Inc., New York, 1995.

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue I January 2025

5. James R. Munkres; Topology; Second Edition, Prentice-Hall of India Private Limited, New Delhi, 2007.
6. Seymour Lipschutz; Theory and Problems of General Topology; Schaum's Series, McGraw-Hill Publications, New York, 1965.
7. Sheldon W. Davis; Topology; McGraw-Hill Higher Education/Walter Rudin Series in Advanced Mathematics, Boston, 2005.
8. Sidney A. Morris; Topology Without Tears; July 24, 2016 Version, From Internet. Link: sidney.morris@gmail.com; and www.sidneymorris.net
9. Royden H.L. and Fitzpatrick P.M.; Real Analysis; PHI Learning Private Limited, 4th Edition, 2012