On the Exponential Diophantine Equation $M_3{}^p + M_5{}^q = r^2$

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Abstract: **Nowadays, researchers are very interested to determine the solution of different Diophantine equations because these equations have many applications in the field of coordinate geometry, trigonometry and applied algebra. These equations help us for finding the integer solution of famous Pythagoras theorem. Finding the solution of Diophantine equations have many challenges for scholars due to absence of generalize methods. In the present paper, authors discussed the exponential Diophantine equation $M_3{}^p + M_5{}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, for existence of its solution.**

Keywords: **Prime number; Diophantine equation; Solution; Mersenne primes, Lucas-Lehmer test.**

Mathematics Subject Classification: 11D61, 11D72, 11D45.

I. INTRODUCTION

Diophantine equations are those equations of theory of numbers which are to be solved in integers. Diophantine equations are classified in two general categories, one is linear Diophantine equations and the other one is non-linear Diophantine equations. Both categories of these equations are very important in theory of numbers and have many important applications in solving the puzzle problems. These equations are very helpful to prove the existence of irrational numbers [4, 6]. Acu [1] studied the Diophantine equation $2^x + 5^y = z^2$ and proved that $\{x = 3, y = 0, z = 3\}$ and $\{x = 2, y = 1, z = 3\}$ are the solutions of this equation. Kumar et al. [2] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [3] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution. Rabago [5] discussed the open problem given by B. Sroysang. He determined that the Diophantine equation $8^x + p^y = z^2$, where x, y, z are positive integers has only three solutions namely $\{x = 1, y = 1, z = 5\}$, $\{x = 2, y = 1, z = 9\}$and $\{x = 3, y = 1, z = 23\}$ for $p = 17$. The Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [7-8]. He proved that these equations have a unique solution which is given by $\{x = 1, y = 0, z = 3\}$. Sroysang [9] proved that the exponential Diophantine equation $31^x + 32^y = z^2$ has no positive integer solution. Aggarwal et al. [10] discussed the existence of solution of Diophantine equation $181^x + 199^y = z^2$. Aggarwal et al. [11] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution.

The main aim of this article is to discuss the exponential Diophantine equation $M_3{}^p + M_5{}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, for existence of its solution.

Preliminaries

Mersenne Numbers [12]: The numbers of the form $M_p = 2^p - 1$ *are called the Mersenne numbers.*

Mersenne Primes [12]: The primes of the form $M_p = 2^p - 1$ *are called the Mersenne primes. Lucas-Lehmer Test [6]: Let* $p \geq 3$. *Then the Mersenne number* M_p *is prime if and only if* $R_{p-1} \equiv 0(mod M_p)$, *where* R_k *is the least residue modulo* M_p *defined by the following recursive relations*

$$\left.\begin{array}{c} R_1 = 4, \\ R_k \equiv R_{k-1}{}^2 - 2(mod M_p), k \geq 2 \end{array}\right\}$$

Using this test, the first ten Mersenne primes are $M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{61}$ *and* M_{89}.

Lemma: 1 The exponential Diophantine equation $M_3{}^p + 1 = r^2$, *where* p, r *are whole numbers and* M_3 *is Mersenne prime, has no solution in whole numbers.*

Proof: Since M_3 is an odd prime so $M_3{}^p$ is an odd number for all whole number p.

$\Rightarrow M_3{}^p + 1 = r^2$ is an even number for all whole number p.

$\Rightarrow r$ is an even number.

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$ (1)

Now, $M_3 \equiv 1(mod3)$

$\Rightarrow M_3{}^p \equiv 1(mod3)$, for whole number p

$\Rightarrow M_3{}^p + 1 \equiv 2(mod3)$, for all whole number p

$\Rightarrow r^2 \equiv 2(mod3)$ (2)

Equation (2) contradicts equation (1). Hence exponential Diophantine equation $M_3{}^p + 1 = r^2$, where p, r are whole numbers and M_3 is Mersenne prime, has no solution in whole number.

Lemma: 2 The exponential Diophantine equation ${M_5}^q + 1 = r^2$, *where* q, r *are whole numbers and* M_5 *is Mersenne prime, has no solution in whole number.*

Proof: Since M_5 is an odd prime so ${M_5}^q$ is an odd number for all whole number q.

$\Rightarrow {M_5}^q + 1 = r^2$ is an even number for all whole number q

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$ (3)

Now, $M_5 \equiv 1(mod3)$

$\Rightarrow {M_5}^q \equiv 1(mod3)$, for all whole number q

$\Rightarrow {M_5}^q + 1 \equiv 2(mod3)$, for all whole number q

$\Rightarrow r^2 \equiv 2(mod3)$ (4)

Equation (4) contradicts equation (3). Hence exponential Diophantine equation ${M_5}^q + 1 = r^2$, where q, r are whole numbers and M_5 is Mersenne prime, has no solution in whole number.

Main Theorem: The exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, has no solution in whole number.

Proof: There are four cases:

Case: 1 If $p = 0$ then the exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$ becomes

$1 + {M_5}^q = r^2$, which has no whole number solution by lemma 2.

Case: 2 If $q = 0$ then the exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$ becomes ${M_3}^p + 1 = r^2$, which has whole number solution by lemma 1.

Case: 3 If p, q are whole numbers, then ${M_3}^p, {M_5}^q$ are odd numbers.

$\Rightarrow {M_3}^p + {M_5}^q = r^2$ is an even number

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)\ or\ r^2 \equiv 1(mod3)$ (5)

Now, $M_3 \equiv 1(mod3)$

$\Rightarrow {M_3}^p \equiv 1(mod3)\ and\ M_5 \equiv 1(mod3)$

$\Rightarrow {M_3}^p \equiv 1(mod3)\ and\ {M_5}^q \equiv 1(mod3)$

$\Rightarrow {M_3}^p + {M_5}^q \equiv 2(mod3)$

$\Rightarrow r^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5). Hence exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, has no whole number solution.

Case: 4 If $p, q = 0$, then ${M_3}^p + {M_5}^q = 1 + 1 = 2 = r^2$, which is impossible because r is a whole number. Hence exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, has no whole number solution.

II. CONCLUSION

In this article, authors successfully discussed the solution of exponential Diophantine equation${M_3}^p + {M_5}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes. They determined that the exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$, where p, q, r are whole numbers, M_3 and M_5 are Mersenne primes, has no whole number solution.

CONFLICT OF INTERESTS

Authors state that this paper has no conflict of interest.

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