On Cosets in Split Extensions of Hypercomplex Numbers

Musyoka David Mwanzia, Lydia Njuguna

Kenyatta University

Abstract: **This paper focuses on the study of some properties of cosets in split extensions of hypercomplex numbers. It is well known that if G is a group and H its subgroup, the cosets of the subgroup H form a partition of the group G. However, this property does not generally hold for loops. This study aims at constructing cyclic subloops of the split extensions of hypercomplex numbers and the corresponding cosets arising from them. It is then shown that the cosets of a cyclic subloop form a partition of the split extension loop i.e. any two right or left cosets of a cyclic subloop are either disjoint or identical. The study uses the Cayley-Dickson and Jonathan Smith doubling processes to construct multiplication tables for the split extensions of hypercomplex numbers. Nim addition is also used to give a general way of generating cyclic subloops and the cosets arising from them. In Loop Theory, only when S is a normal subloop of Lwill the left and right cosets of S coincide, these cosets form a loopL/Scalled the quotient or factor loop whose multiplication is defined by $(a \cdot S) \cdot (b \cdot S) = (a \cdot b) \cdot S, \forall a, b \in L$. In this workwe use cyclic normal subloops of split extensions of hypercomplex numbers to construct quotient loops, and show that the multiplication of the elements in the quotient loop formed can also be carried out by considering the Nim addition of the subscripts of the individual elements. The complex split extension forms a group and hence it remains trivial to work on the same. Though the authors have also carried out the same process on the sedenion split extensions, the present paper focusesmainly on the quaternion and octonion split extension.**

AMS Subject Classification: 20B05

Key words: **Loop, Hypercomplex numbers, split extension, Cayley – Dickson, Quotient loop, Coset decomposition**

I. INTRODUCTION

In Mathematics, a coset is a set made of all the products obtained from multiplication of every element of a subgroup H in turn by one element of the group G that contain the subgroup H. Multiplication of an element of a group by the subgroup from the left gives rise to a left coset while multiplication from the right gives rise to a right coset. A coset may not necessarily be a subgroup of the group.

Cosets form the basis of this study. They play a very crucial role in proofs of some of the most basic results in Group Theory, for instance, in the proof of the Lagrange's Theorem(Kinyon M, Pula K and Vojtechovsky P, 2012). They are also used in construction of quotient loops. Recently, Mathematicians have focused their attention on the study of cosets in Loop Theory. Michael Kinyon, Kyle Pula and Petr Vojtechovsky studied the properties of cosets in Antiautomorphic loops and Bol loops(Kinyon M, Pula K and Vojtechovsky P, 2012). They showed that any two left cosets of a subloop S of a left automorphic Moufang loop were either disjoint or intersect in a set whose number of elements equals that of some subloop of S.Ales Drapal and Terry Griggs gave a complete answer tothe question of when the cosets of a Steiner subloop Spartition the loop(Drapal A., Griggs T. S, 2016). They concluded that this happens if and only if the Steiner loop can be formed by a union of subloops of order$2|S|$, any two of which intersect in S.

The Cayley-Dickson doubling Process

The Cayley- Dickson doubling process starting from real numbers successively yields the complex numbers of dimension 2, quaternions of dimension 4, octonions of dimension 8, and sedenions of dimension 16(Smith, W. D, 2004). Each algebra contains the previous one as a sub algebra. Different doubling processes for obtaining sedenions from octonions and general 2^n-ons from 2^{n-1}-ons have been developed.

A complex number can be written in the form (a, b) where$a, b \in \mathbb{R}$. Doubling complex numbers by using the Cayley-Dickson process gives rise to a 2^2 – dimensional quaternion algebra. The multiplication is defined as follows:

$$(a, b)(c, d) = (ac - b\bar{d}, b\bar{c} + ad) \qquad (1)$$

The 2^2 – dimensional quaternion algebra$\mathbb{H}$ has a basis $1, i, j$ and k. Therefore, if $q \in \mathbb{H}$, it can be written as,$q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$.

The multiplication for quaternions can be expressed as a rule i. e $i^2 = j^2 = k^2 = ijk = -1$ which implies that, $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$.

Next we have the 2^3-dimensional octonions algebra obtained by forming pairs (a, b) where $a, b \in \mathbb{H}$ and multiplication carried out as in complex numbers. In general, this process continues giving rise to 2^n-dimensional hypercomplex numbers from a pair of 2^{n-1}-dimensional hypercomplex numbers.

Multiplication of the split extension elements

The elements of $L \rtimes S^0$ are encoded as pairs $(a, b)$with $a \in L$ and$b \in S^0 = \{1, -1\}$. The multiplication of these elements will be done using the Jonathan Smith formula (Smith, J. D. H, 1995) given below:

i. $(x,1)(y,1) = (xy,1)$
ii. $(x,1)(y,-1) = (yx,-1)$
iii. $(x,-1)(y,1) = (x\bar{y},-1)$
iv. $(x,-1)(y,-1) = (-x\bar{y},1)$ (2)

II. PRELIMINARIES

Definition 1: Groupoid

A groupoid $(Q,\ \cdot)$ is a non-empty set Q which is closed under the binary operation$(\cdot)$ i.e.

$\forall x, y \in Q\ \exists\ z \in Q$ such that $x \cdot y = z$.

Definition 2: Quasigroup

A quasigroup is a groupoid $(Q,\ \cdot)$ such that $\forall\ a, b \in Q$ the equations $ax = b$, and $ya = b$ have unique solutions $x, y \in Q$ respectively.

Definition 3: Loop

A Loop is a quasigroup $(Q,\ \cdot)$ with a neutral element $e \in Q$ such that $ex = xe = x\ \forall x \in Q$. A loop with the associative property forms a group i.e. in Loop Theory a group is simply an associative loop.

Definition 4: Subloop

A subloop is a non-empty subset $S \subseteq (Q,\ \cdot)$ denoted by $S \leq Q$ such that$(S,\ \cdot)$ is a loop in its own right.

Definition 5: Left coset

For a loop Q, a subloop $S \leq Q$, and $x \in Q$, then the subset $xS = \{xs: s \in S\} \subseteq Q$ is the left cosetof S containing x.

Definition 6: Right coset

For a loop Q, a subloop $S \leq Q$, and $x \in Q$, then the subset $Sx = \{sx: s \in S\} \subseteq Q$ is the right coset of S containing$x$.

Definition 7: Commutant

The commutant, $C(Q)$, of a loop Q is the set of those elements $c \in Q$which commute with each element in the loop. That is, $C(Q) = \{c \in Q :\ \forall x \in Q, cx = xc\}$

Defitnition 8: Decomposition property

A loop Q has left (respectively right) coset decomposition modulo S if the set of all left (respectively right) cosets modulo S is a partition of Q(Drapal A., Griggs T. S, 2016).

Definition 9: Bol loop

A Bol loop is loop $(Q,\ \cdot)$satisfying the left or right Bol laws i.e. $\forall\ x, y, z \in Q$:

(i) $(x \cdot yx)z = x(y \cdot xz)$ Left Bol law,
(ii) $z(xy \cdot x) = (zx \cdot y)x)$ Right Bol law.

Definition 10: Moufang loop

A loop L is called a Moufang loop if it satisfies any of the following equivalent identities:

(i) $xy.zx = (x.yz)x$,
(ii) $x(y.xz) = (xy.x)z$,
(iii) $x(y.zy) = (xy.z)y$ for all $x, y, z \in L$.

Definition 11: Steiner loop

A Steiner loop or a sloop is a groupoid $(L,\cdot, e)$, where $(\cdot)$ is a binary operation and e is a constant satisfying the identities:

(i) $e \cdot x = x$,
(ii) $x \cdot y = y \cdot x$,
(iii) $x \cdot (x \cdot y) = y$ for all $x, y \in L$.

Definition 12: Antiautomorphic loop

A loop L is said to be an antiautomorphic loop if it satisfies the following property

$\overline{xy} = \bar{y}\bar{x} \forall\ x, y \in L$.

Definition 13: Normal Subloop

A subloop S of a loop L is said to be normal if for all $a, b \in L$, the following holds:

$$(a \cdot b) \cdot S = a \cdot (b \cdot S) = a \cdot (S \cdot b)$$

Note that the equality of the first two is not guaranteed because we do not assume the loop to be associative.

Definition 14: Quotient loop

Let $(L,\cdot)$ be a loop and S a normal subloop of L. The quotient loop L/S is defined as the following loop:

1. The set of elements of L/S is the set of left cosets of S,i.e. subsets of the form $a \cdot S$, with $a \in L$.
2. The multiplication is defined by $(a \cdot S) \cdot (b \cdot S) = (a \cdot b) \cdot S$ and is well defined and follows from the definition of a normal subloop.

Nim Addition

Nim addition gives a convenient way of defining addition in $\mathbb{Z}^+$ to make it a field of characteristic two. It is carried out by first writing the individual numbers in binary form and then adding without carrying over.

Rules for Nim-addition

i. The Nim sum of a number of distinct 2-powers is their ordinary sum.
ii. The Nim sum of two equal numbers is zero.

In this study we use Nim addition to:

i. Generalize the construction of a cyclic subloop of the split extensions of hypercomplex numbers.
ii. Give a general way of generating left and right cosets of a subloop.
iii. Show that the multiplication of the elements of a quotient loop is similar to Nim addition of the subscripts of the individual elements.

Some properties of Nim addition:

(a) $\alpha \oplus \beta = \beta \oplus \alpha$
(b) $\alpha \oplus 0 = 0 \oplus \alpha = \alpha$
(c) $\alpha \oplus \alpha = 0$
(d) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$

III. RESULTS AND DISCUSSION

Quaternion slit extensions

Quaternions $\mathbb{H}$form a four-dimensional algebra with the basis $1, i, j,$ and k. The multiplication of these basis elements is defined by:

$i^2 = j^2 = k^2 = ijk = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i,$ and $ik = -j$.

The quaternion split extensions $\mathbb{H} \rtimes S^\circ$ forms a set of order 16. The basal elements of $\mathbb{H} \rtimes S^\circ$ are defined as follows

$\beta_0 = (1,1)$, $\beta_1 = (i, 1)$, $\beta_2 = (j, 1)$, $\beta_3 = (k, 1)$, $\beta_4 = (-1, -1), \beta_5 = (-i, -1)$, $\beta_6 = (-j, -1)$, $\beta_7 = (-k, -1), \beta_8 = (-1,1)$, $\beta_9 = (-i, 1)$, $\beta_{10} = (-j, 1)$, $\beta_{11} = (-k, 1)$, $\beta_{12} = (1, -1)$,

$\beta_{13} = (i, -1)$, $\beta_{14} = (j, -1)$, $\beta_{15} = (k, -1)$

The multiplication of these elements is given by the following table:

Table 1: Multiplication of quaternion split extensions (Magero F.B, 2007)

	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}
β_0	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}
β_1	β_1	β_8	β_3	β_{10}	β_5	β_{12}	β_{15}	β_6	β_9	β_0	β_{11}	β_2	β_{13}	β_4	β_7	β_{14}
β_2	β_2	β_{11}	β_8	β_1	β_6	β_7	β_{12}	β_{13}	β_{10}	β_3	β_0	β_9	β_{14}	β_{15}	β_4	β_5
β_3	β_3	β_2	β_9	β_8	β_7	β_{14}	β_5	β_{12}	β_{11}	β_{10}	β_1	β_0	β_{15}	β_6	β_{13}	β_4
β_4	β_4	β_{13}	β_{14}	β_{15}	β_8	β_1	β_2	β_3	β_{12}	β_5	β_6	β_7	β_0	β_9	β_{10}	β_{11}
β_5	β_5	β_4	β_{15}	β_6	β_9	β_8	β_3	β_{10}	β_{13}	β_{12}	β_7	β_{14}	β_1	β_0	β_{11}	β_2
β_6	β_6	β_7	β_4	β_{13}	β_{10}	β_{11}	β_8	β_1	β_{14}	β_{15}	β_{12}	β_5	β_2	β_3	β_0	β_9
β_7	β_7	β_{14}	β_5	β_4	β_{11}	β_2	β_9	β_8	β_{15}	β_6	β_{13}	β_{12}	β_3	β_{10}	β_1	β_0
β_8	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
β_9	β_9	β_0	β_{11}	β_2	β_{13}	β_4	β_7	β_{14}	β_1	β_8	β_3	β_{10}	β_5	β_{12}	β_{15}	β_6
β_{10}	β_{10}	β_3	β_0	β_9	β_{14}	β_{15}	β_4	β_5	β_2	β_{11}	β_8	β_1	β_6	β_7	β_{12}	β_{13}
β_{11}	β_{11}	β_{10}	β_1	β_0	β_{15}	β_6	β_{13}	β_4	β_3	β_2	β_9	β_8	β_7	β_{14}	β_5	β_{12}
β_{12}	β_{12}	β_5	β_6	β_7	β_0	β_9	β_{10}	β_{11}	β_4	β_{13}	β_{14}	β_{15}	β_8	β_1	β_2	β_3
β_{13}	β_{13}	β_{12}	β_7	β_{14}	β_1	β_0	β_{11}	β_2	β_5	β_4	β_{15}	β_6	β_9	β_8	β_3	β_{10}
β_{14}	β_{14}	β_{15}	β_{12}	β_5	β_2	β_3	β_0	β_9	β_6	β_7	β_4	β_{13}	β_{10}	β_{11}	β_8	β_1
β_{15}	β_{15}	β_6	β_{13}	β_{12}	β_3	β_{10}	β_1	β_0	β_7	β_{14}	β_5	β_4	β_{11}	β_2	β_9	β_8

Observations

1. Each element appears only once in every row and every column. Thus any two elements define a third one uniquely i.e.$\mathbb{H} \rtimes S^\circ$ is a quasigroup.
2. β_0 is the two-sided identity i.e. $\beta_0\beta_i = \beta_i\beta_0 = \beta_i$ for all $i = \{0,1,2, \dots, 15\}$, and hence $\mathbb{H} \rtimes S^\circ$ forms a loop.
3. $\mathbb{H} \rtimes S^\circ$ is not associative since
$$(\beta_i\beta_j)\beta_k \neq \beta_i(\beta_j\beta_k) \; \forall i, j, k = \{0,1,2, \dots, 15\}$$

Example 3.2.1

$$(\beta_9\beta_{11})\beta_{12} = \beta_{10}.\beta_{12} = \beta_6 \neq \beta_9(\beta_{11}\beta_{12}) = \beta_9.\beta_7 = \beta_{14}$$
$$\{(-i, 1)(-k, 1)\}(1, -1) = (ik, 1)(1, -1) = (-j, -1)$$
$$\neq (-i, 1)\{(-k, 1)(1, -1)\}$$
$$= (-i, 1)(-k, -1) = (ki, -1) = (j, -1)$$

Cyclic subloops of quaternion split extensions

In this section we construct cyclic subloops generated by each element in quaternion split extensions. For example, the cyclic subloop generated by β_1, which is denoted by $\langle\beta_1\rangle$ can be constructed as follows.

$\langle\beta_1\rangle^1 = (i, 1)^1 = (i, 1) = \beta_1$

$\langle\beta_1\rangle^2 = (i, 1)(i, 1) = (-1,1) = \beta_8$

$\langle\beta_1\rangle^3 = (-1,1)(i, 1) = (-i, 1) = \beta_9$

$\langle\beta_1\rangle^4 = (-i, 1)(i, 1) = (1,1) = \beta_0$

Thus,

$\langle\beta_1\rangle = \{\beta_1, \beta_8, \beta_9, \beta_0\} = \langle\beta_9\rangle$

Similarly, we can construct other cyclic subloops generated by other elements,

1. $\langle\beta_2\rangle = \{\beta_2, \beta_8, \beta_{10}, \beta_0\} = \langle\beta_{10}\rangle$
2. $\langle\beta_3\rangle = \{\beta_3, \beta_8, \beta_{11}, \beta_0\} = \langle\beta_{11}\rangle$
3. $\langle\beta_4\rangle = \{\beta_4, \beta_8, \beta_{12}, \beta_0\} = \langle\beta_{12}\rangle$
4. $\langle\beta_5\rangle = \{\beta_5, \beta_8, \beta_{13}, \beta_0\} = \langle\beta_{13}\rangle$
5. $\langle\beta_6\rangle = \{\beta_6, \beta_8, \beta_{14}, \beta_0\} = \langle\beta_{14}\rangle$
6. $\langle\beta_7\rangle = \{\beta_7, \beta_8, \beta_{15}, \beta_0\} = \langle\beta_{15}\rangle$
7. $\langle\beta_8\rangle = \{\beta_8, \beta_0\}$

Remark 3.2.1.1 In this case,$\langle\beta_8\rangle = \{\beta_8,\ \beta_0\}$ is the commutant of $\mathbb{H} \rtimes S^\circ$ with $|\langle\beta_8\rangle| = 2$

Theorem 3.2.1Let$\langle\beta_i\rangle$ be a cyclic subloop of $\mathbb{H} \rtimes S^\circ$, then:

1. The elements of the cyclic subloop can be generated using Nim addition as follows:

$$\langle\beta_i\rangle = \{\beta_i, \beta_c, \beta_{c\oplus i}, \beta_0\}$$

where, $\beta_i$is the generating element, for all $i \in \{0,1,\dots,15\}$, and β_c is the non-identity element in the commutant, $C(\mathbb{H} \rtimes S^\circ)$

2. $\langle\beta_i\rangle = \langle\beta_{i\ \oplus\ c}\rangle\ \forall\ \beta_i \in \mathbb{H} \rtimes S^0$ and c the subscript for the non-identity element in $C(\mathbb{H} \rtimes S^\circ)$ provided that $i \neq 0$

Coset decomposition

We now construct distinct left and right cosets of some cyclic subloops in section 3. 2. 1 above. *Examples 3.2.2.1*

1. Let S = $\langle\beta_8\rangle = \{\beta_8, \beta_0\}$ then the left cosets of S are as follows:

$$\beta_1 S = \beta_1\{\beta_8, \beta_0\} = \{\beta_9, \beta_1\} = \{\beta_8, \beta_0\}\beta_1 = S\beta_1$$

$$\beta_2 S = \beta_2\{\beta_8, \beta_0\} = \{\beta_{10}, \beta_2\} = \{\beta_8, \beta_0\}\beta_2 = S\beta_2$$

$$\beta_3 S = \beta_3\{\beta_8, \beta_0\} = \{\beta_{11}, \beta_3\} = \{\beta_8, \beta_0\}\beta_3 = S\beta_3$$

$$\beta_4 S = \beta_4\{\beta_8, \beta_0\} = \{\beta_{12}, \beta_4\} = \{\beta_8, \beta_0\}\beta_4 = S\beta_4$$

$$\beta_5 S = \beta_5\{\beta_8, \beta_0\} = \{\beta_{13}, \beta_5\} = \{\beta_8, \beta_0\}\beta_5 = S\beta_5$$

$$\beta_6 S = \beta_6\{\beta_8, \beta_0\} = \{\beta_{14}, \beta_6\} = \{\beta_8, \beta_0\}\beta_6 = S\beta_6$$

$$\beta_7 S = \beta_7\{\beta_8, \beta_0\} = \{\beta_{15}, \beta_7\} = \{\beta_8, \beta_0\}\beta_7 = S\beta_7$$

2. If $S = \langle\beta_1\rangle = \{\beta_1, \beta_8, \beta_9, \beta_0\} = \langle\beta_9\rangle$ then

$$\beta_2 S = \beta_2\{\beta_1, \beta_8, \beta_9, \beta_0\} = \{\beta_{11}, \beta_{10}, \beta_3, \beta_2\},$$

$$\beta_4 S = \beta_4\{\beta_1, \beta_8, \beta_9, \beta_0\} = \{\beta_{13}, \beta_{12}, \beta_5, \beta_4\},$$

$$\beta_6 S = \beta_6\{\beta_1, \beta_8, \beta_9, \beta_0\} = \{\beta_7, \beta_{14}, \beta_{15}, \beta_6\}$$

$$S\beta_2 = \{\beta_1, \beta_8, \beta_9, \beta_0\}\beta_2 = \{\beta_3, \beta_{10}, \beta_{11}, \beta_2\},$$

$$S\beta_4 = \{\beta_1, \beta_8, \beta_9, \beta_0\}\beta_4 = \{\beta_5, \beta_{12}, \beta_{13}, \beta_4\},$$

$$S\beta_6 = \{\beta_1, \beta_8, \beta_9, \beta_0\}\beta_6 = \{\beta_{15}, \beta_{14}, \beta_7, \beta_6\}$$

3. If $S = \langle\beta_2\rangle = \{\beta_2, \beta_8, \beta_{10}, \beta_0\} = \langle\beta_{10}\rangle$ then,

$$\beta_1 S = \beta_1\{\beta_2, \beta_8, \beta_{10}, \beta_0\} = \{\beta_3, \beta_9, \beta_{11}, \beta_1\},$$

$$\beta_4 S = \beta_4\{\beta_2, \beta_8, \beta_{10}, \beta_0\} = \{\beta_{14}, \beta_{12}, \beta_6, \beta_4\},$$

$$\beta_5 S = \beta_5\{\beta_2, \beta_8, \beta_{10}, \beta_0\} = \{\beta_{15}, \beta_{13}, \beta_7, \beta_5\}$$

$$S\beta_1 = \{\beta_2, \beta_8, \beta_{10}, \beta_0\}\beta_1 = \{\beta_{11}, \beta_9, \beta_3, \beta_1\},$$

$$S\beta_4 = \{\beta_2, \beta_8, \beta_{10}, \beta_0\}\beta_4 = \{\beta_6, \beta_{12}, \beta_{14}, \beta_4\},$$

$$S\beta_5 = \{\beta_2, \beta_8, \beta_{10}, \beta_0\}\beta_5 = \{\beta_7, \beta_{14}, \beta_{15}, \beta_5\}$$

Following the same method, the left and right cosets generated by the other cyclic subloops of the quaternion split extensions can be obtained.

Theorem 3.2.2 Let $S = \langle\beta_i\rangle = \{\beta_i, \beta_c, \beta_{c\oplus i}, \beta_0\}$ be a cyclic subloop of $\mathbb{H} \rtimes S^\circ$,to get the left cosets of S, we choose an element $\beta_j \in \mathbb{H} \rtimes S^\circ$such that $\beta_j \notin S$ and carry out Nim addition as follows:

$$\beta_j S = \{\beta_{(j\oplus i)}, \beta_{(j\oplus c)}, \beta_{j\oplus(c\oplus i)}, \beta_{(j\oplus 0)}\}$$

On the other hand, to get the right cosets of $S = \langle\beta_i\rangle = \{\beta_i, \beta_c, \beta_{c\oplus i}, \beta_0$, we have,

$$S\beta_j = \{\beta_{(i\oplus j)}, \beta_{(c\oplus j)}, \beta_{(c\oplus i)\oplus j}, \beta_{(0\oplus j)}\}$$

Example 3.2.2.2

Let $S = \langle\beta_7\rangle = \{\beta_7, \beta_8, \beta_{15}, \beta_0\}$ then the subscripts for the elements of $\beta_3 S$ are obtained by Nim addition as follows:

$$\beta_3 S \quad = \beta_3\{\beta_7, \beta_8, \beta_{15}, \beta_0\}$$

$$= \{\beta_3\beta_7,\ \beta_3\beta_8,\ \beta_3\beta_{15},\ \beta_3\beta_0\}.$$

$$= \{\beta_{3\oplus 7}, \beta_{3\oplus 8}, \beta_{3\oplus 15}, \beta_{3\oplus 0}\}$$

$$= \{\beta_4, \beta_{11}, \beta_{12}, \beta_3\}$$

Conclusions

Given$\forall\ \beta_i, \beta_j \in \mathbb{H} \rtimes S^\circ, i, j \in \{0,\dots,15\}$

(i) $\beta_i \in \beta_i S$ or $\beta_i \in S\beta_i$
(ii) $\beta_i S = \beta_j S$ and $\beta_i S \cap \beta_j S = \emptyset$or $S\beta_i = S\beta_j$ and $S\beta_i \cap S\beta_j = \emptyset$
(iii) $|\beta_i S| = |\beta_j S|\ or |S\beta_i| = |S\beta_j|$

Therefore, the left or right cosets of a cyclic subloop$S$ partition $\mathbb{H} \rtimes S^\circ$ into equivalence classes under the relation $\beta_i \sim \beta_j$.

Quotient loops arising from quaternion split extension

In this section we construct quotient loops using cyclic normal subloops of the quaternion split extensions. First, we consider a cyclic subloop of order 2 given by $S = \langle\beta_8\rangle = \{\beta_8, \beta_0\}$. The left cosets of S are:

$$\beta_0 S = \beta_0\{\beta_8, \beta_0\} = \{\beta_8, \beta_0\}, \beta_1 S = \beta_1\{\beta_8, \beta_0\} = \{\beta_9, \beta_1\},$$

$$\beta_2 S = \beta_2\{\beta_8, \beta_0\} = \{\beta_{10}, \beta_2\}, \beta_3 S = \beta_3\{\beta_8, \beta_0\} = \{\beta_{11}, \beta_3\},$$

$$\beta_4 S = \beta_4\{\beta_8, \beta_0\} = \{\beta_{12}, \beta_4\}, \beta_5 S = \beta_5\{\beta_8, \beta_0\} = \{\beta_{13}, \beta_5\},$$

$\beta_6 S = \beta_6\{\beta_8, \beta_0\} = \{\beta_{14}, \beta_6\}$,and $\beta_7 S = \beta_7\{\beta_8, \beta_0\} = \{\beta_{15}, \beta_7\}$.

The elements of the quotient loop$\mathbb{H} \rtimes S^\circ/S$ are $\beta_0 S, \beta_1 S, \beta_2 S, \beta_3 S, \beta_4 S, \beta_5 S, \beta_6 S$ and $\beta_7 S$. The multiplication of these elements is given by the following table.

Table 2: Multiplication of the elements of $\mathbb{H} \rtimes S^\circ/S$

	β_0S	β_1S	β_2S	β_3S	β_4S	β_5S	β_6S	β_7S
β_0S	β_0S	β_1S	β_2S	β_3S	β_4S	β_5S	β_6S	β_7S
β_1S	β_1S	β_0S	β_3S	β_2S	β_5S	β_4S	β_7S	β_6S
β_2S	β_2S	β_3S	β_0S	β_1S	β_6S	β_7S	β_4S	β_5S
β_3S	β_3S	β_2S	β_1S	β_0S	β_7S	β_6S	β_5S	β_4S
β_4S	β_4S	β_5S	β_6S	β_7S	β_0S	β_1S	β_2S	β_3S
β_5S	β_5S	β_4S	β_7S	β_6S	β_1S	β_0S	β_3S	β_2S
β_6S	β_6S	β_7S	β_4S	β_5S	β_2S	β_3S	β_0S	β_1S
β_7S	β_7S	β_6S	β_5S	β_4S	β_3S	β_2S	β_1S	β_0S

To construct the Nim addition table of the subscripts of the elements of $\mathbb{H} \rtimes S^\circ/S$i.e. 0, 1, 2, 3, 4, 5, 6 and 7refer rules for Nim addition in the preliminaries section.

Next we construct a quotient loop using a cyclic normal subloop $M = \langle\beta_2\rangle = \{\beta_2, \beta_8, \beta_{10}, \beta_0\}$ of order 4. The left cosets of M are:

$\beta_0 M = \{\beta_2, \beta_8, \beta_{10}, \beta_0\}$, $\beta_1 M = \{\beta_3, \beta_9, \beta_{11}, \beta_1\}$,

$\beta_4 M = \{\beta_{14}, \beta_{12}, \beta_6, \beta_4\}$ and$\beta_5 M = \{\beta_{15}, \beta_{13}, \beta_7, \beta_5\}$.

Therefore, the elements of the quotient loop $\mathbb{H} \rtimes S^\circ/M$ are,$\beta_0 M$, $\beta_1 M$, $\beta_4 M$ and $\beta_5 M$. The multiplication of these elements is given by the following table.

Table 3: Multiplication of the elements of $\mathbb{H} \rtimes S^\circ/M$

	β_0M	β_1M	β_4M	β_5M
β_0M	β_0M	β_1M	β_4M	β_5M
β_1M	β_1M	β_0M	β_5M	β_4M
β_4M	β_4M	β_5M	β_0M	β_1M
β_5M	β_5M	β_4M	β_1M	β_0M

To get the Nim addition table of elements 0, 1, 4 and 5 refer rules for nim addition in the preliminaries section.

From the above tables we observe that $\forall i, j, k \in \{0,1, \dots ,15\}$ and S a cyclic subloop of $\mathbb{H} \rtimes S^\circ$

(i) $\beta_i S . \beta_j S = \beta_{i \oplus j} S$

(ii) $\beta_i S . \beta_j S = \beta_j S . \beta_i S$ since $i \oplus j = j \oplus i$

(iii) $\beta_i S . \beta_i S = \beta_0 S$ since $i \oplus i = 0$

(iv) $\beta_i S(\beta_j S \beta_k S) = (\beta_i S \beta_j S)\beta_k S$ since $i \oplus (j \oplus k) = (i \oplus j) \oplus k$

For a subloop S of order 2 the number of distinct cosets is given by; $\frac{16}{2} = 8$.

We again notice that the subscripts of the elements of a quotient loop formed from quaternion split extensions are closed under Nim addition. The multiplication of the elements of $\mathbb{H} \rtimes S^\circ/S$can therefore be achieved by using Nim addition.

Octonion split extensions

Octonions $\mathbb{O}$form an 2^3-dimension algebra whose basis elements are,$\{e_0 = (1,0), e_1 = (i, 0),$

$e_2 = (j, 0), e_3 = (k, 0), e_4 = (0,1), e_5 = (0, i), e_6 = (0, j), e_7 = (0, k)\}$. The multiplication of these elements will be carried out using the Cayley Dickson process.

The Octonion Split Extensions form a set of order 32. Its elements are of the form (x, y) where $x \in \mathbb{O}$ and $y \in S^\circ$. The elements of $\mathbb{O} \rtimes S^\circ$ are thus defined as follows:

$$\mu_0 = (e_0, 1), \mu_1 = (e_1, 1), \mu_2 = (e_2, 1), \mu_3 = (e_3, 1), \mu_4 = (e_4, 1), \mu_5 = (e_5, 1),$$

$$\mu_6 = (e_6, 1), \mu_7 = (e_7, 1), \mu_8 = (-e_0, 1), \mu_9 = (-e_1, 1), \mu_{10} = (-e_2, 1), \mu_{11} = (-e_3, 1),$$

$$\mu_{12} = (-e_4, 1), \mu_{13} = (-e_5, 1), \mu_{14} = (-e_6, 1), \mu_{15} = (-e_7, 1), \mu_{16} = (e_0, -1),$$

$$\mu_{17} = (e_1, -1), \mu_{18} = (e_2, -1), \mu_{19} = (e_3, -1), \mu_{20} = (e_4, -1), \mu_{21} = (e_5, -1),$$

$$\mu_{22} = (e_6, -1), \mu_{23} = (e_7, -1), \mu_{24} = (-e_0, -1), \mu_{25} = (-e_1, -1), \mu_{26} = (-e_2, -1),$$

$\mu_{27} = (-e_3, -1), \mu_{28} = (-e_4, -1), \mu_{29} = (-e_5, -1)$, s$\mu_{30} = (-e_6, -1), \mu_{31} = (-e_7, -1)$

The multiplication table of these elements is given by the following table:

Table4. Multiplication of octonion split extensions (Magero F.B, 2007)

	μ_{0}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{8}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}
μ_{0}	μ_{0}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{8}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}
μ_{1}	μ_{1}	μ_{8}	μ_{3}	μ_{10}	μ_{5}	μ_{12}	μ_{15}	μ_{6}	μ_{9}	μ_{0}	μ_{11}	μ_{2}	μ_{13}	μ_{4}	μ_{7}	μ_{14}	μ_{17}	μ_{24}	μ_{27}	μ_{18}	μ_{29}	μ_{20}	μ_{23}	μ_{30}	μ_{25}	μ_{16}	μ_{19}	μ_{26}	μ_{21}	μ_{28}	μ_{31}	μ_{22}
μ_{2}	μ_{2}	μ_{11}	μ_{8}	μ_{1}	μ_{6}	μ_{7}	μ_{12}	μ_{13}	μ_{10}	μ_{3}	μ_{0}	μ_{9}	μ_{14}	μ_{15}	μ_{4}	μ_{5}	μ_{18}	μ_{19}	μ_{24}	μ_{25}	μ_{30}	μ_{31}	μ_{20}	μ_{21}	μ_{26}	μ_{27}	μ_{16}	μ_{17}	μ_{22}	μ_{23}	μ_{28}	μ_{29}
μ_{3}	μ_{3}	μ_{2}	μ_{9}	μ_{8}	μ_{7}	μ_{14}	μ_{5}	μ_{12}	μ_{11}	μ_{10}	μ_{1}	μ_{0}	μ_{15}	μ_{6}	μ_{13}	μ_{4}	μ_{19}	μ_{26}	μ_{17}	μ_{24}	μ_{31}	μ_{22}	μ_{29}	μ_{20}	μ_{27}	μ_{18}	μ_{25}	μ_{16}	μ_{23}	μ_{30}	μ_{21}	μ_{28}
μ_{4}	μ_{4}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{12}	μ_{5}	μ_{6}	μ_{7}	μ_{0}	μ_{9}	μ_{10}	μ_{11}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{16}	μ_{17}	μ_{18}	μ_{19}
μ_{5}	μ_{5}	μ_{4}	μ_{15}	μ_{6}	μ_{9}	μ_{8}	μ_{11}	μ_{2}	μ_{13}	μ_{12}	μ_{7}	μ_{14}	μ_{1}	μ_{0}	μ_{3}	μ_{10}	μ_{21}	μ_{28}	μ_{23}	μ_{30}	μ_{17}	μ_{24}	μ_{19}	μ_{26}	μ_{29}	μ_{20}	μ_{31}	μ_{22}	μ_{25}	μ_{16}	μ_{27}	μ_{18}
μ_{6}	μ_{6}	μ_{7}	μ_{4}	μ_{13}	μ_{10}	μ_{3}	μ_{8}	μ_{9}	μ_{14}	μ_{15}	μ_{12}	μ_{5}	μ_{2}	μ_{11}	μ_{0}	μ_{1}	μ_{22}	μ_{31}	μ_{28}	μ_{21}	μ_{18}	μ_{27}	μ_{24}	μ_{17}	μ_{30}	μ_{23}	μ_{20}	μ_{29}	μ_{26}	μ_{19}	μ_{16}	μ_{25}
μ_{7}	μ_{7}	μ_{14}	μ_{5}	μ_{4}	μ_{11}	μ_{10}	μ_{1}	μ_{8}	μ_{15}	μ_{6}	μ_{13}	μ_{12}	μ_{3}	μ_{2}	μ_{9}	μ_{0}	μ_{23}	μ_{22}	μ_{29}	μ_{28}	μ_{19}	μ_{18}	μ_{25}	μ_{24}	μ_{31}	μ_{30}	μ_{21}	μ_{20}	μ_{27}	μ_{26}	μ_{17}	μ_{16}
μ_{8}	μ_{8}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}
μ_{9}	μ_{9}	μ_{0}	μ_{11}	μ_{2}	μ_{13}	μ_{4}	μ_{7}	μ_{14}	μ_{1}	μ_{8}	μ_{3}	μ_{10}	μ_{5}	μ_{12}	μ_{15}	μ_{6}	μ_{25}	μ_{16}	μ_{19}	μ_{26}	μ_{21}	μ_{28}	μ_{31}	μ_{22}	μ_{17}	μ_{24}	μ_{27}	μ_{18}	μ_{29}	μ_{20}	μ_{23}	μ_{30}
μ_{10}	μ_{10}	μ_{3}	μ_{0}	μ_{9}	μ_{14}	μ_{15}	μ_{4}	μ_{5}	μ_{2}	μ_{11}	μ_{8}	μ_{1}	μ_{6}	μ_{7}	μ_{12}	μ_{13}	μ_{26}	μ_{27}	μ_{16}	μ_{17}	μ_{22}	μ_{23}	μ_{28}	μ_{29}	μ_{18}	μ_{19}	μ_{24}	μ_{25}	μ_{30}	μ_{31}	μ_{20}	μ_{21}
μ_{11}	μ_{11}	μ_{10}	μ_{1}	μ_{0}	μ_{15}	μ_{6}	μ_{13}	μ_{4}	μ_{3}	μ_{2}	μ_{9}	μ_{8}	μ_{7}	μ_{14}	μ_{5}	μ_{12}	μ_{27}	μ_{18}	μ_{25}	μ_{16}	μ_{23}	μ_{30}	μ_{21}	μ_{28}	μ_{19}	μ_{26}	μ_{17}	μ_{24}	μ_{31}	μ_{22}	μ_{29}	μ_{20}
μ_{12}	μ_{12}	μ_{5}	μ_{6}	μ_{7}	μ_{0}	μ_{9}	μ_{10}	μ_{11}	μ_{4}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}
μ_{13}	μ_{13}	μ_{12}	μ_{7}	μ_{14}	μ_{1}	μ_{0}	μ_{3}	μ_{10}	μ_{5}	μ_{4}	μ_{15}	μ_{6}	μ_{9}	μ_{8}	μ_{11}	μ_{2}	μ_{29}	μ_{20}	μ_{31}	μ_{22}	μ_{25}	μ_{16}	μ_{27}	μ_{18}	μ_{21}	μ_{28}	μ_{23}	μ_{30}	μ_{17}	μ_{24}	μ_{19}	μ_{26}
μ_{14}	μ_{14}	μ_{15}	μ_{12}	μ_{5}	μ_{2}	μ_{11}	μ_{0}	μ_{1}	μ_{6}	μ_{7}	μ_{4}	μ_{13}	μ_{10}	μ_{3}	μ_{8}	μ_{9}	μ_{30}	μ_{23}	μ_{20}	μ_{29}	μ_{26}	μ_{19}	μ_{16}	μ_{25}	μ_{22}	μ_{31}	μ_{28}	μ_{21}	μ_{18}	μ_{27}	μ_{24}	μ_{17}
μ_{15}	μ_{15}	μ_{6}	μ_{13}	μ_{12}	μ_{3}	μ_{2}	μ_{9}	μ_{0}	μ_{7}	μ_{14}	μ_{5}	μ_{4}	μ_{11}	μ_{10}	μ_{1}	μ_{8}	μ_{31}	μ_{30}	μ_{21}	μ_{20}	μ_{27}	μ_{26}	μ_{17}	μ_{16}	μ_{23}	μ_{22}	μ_{29}	μ_{28}	μ_{19}	μ_{18}	μ_{25}	μ_{24}
μ_{16}	μ_{16}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{24}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{0}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}
μ_{17}	μ_{17}	μ_{16}	μ_{27}	μ_{18}	μ_{29}	μ_{20}	μ_{23}	μ_{30}	μ_{25}	μ_{24}	μ_{19}	μ_{26}	μ_{21}	μ_{28}	μ_{31}	μ_{22}	μ_{9}	μ_{8}	μ_{3}	μ_{10}	μ_{5}	μ_{12}	μ_{15}	μ_{6}	μ_{1}	μ_{0}	μ_{11}	μ_{2}	μ_{13}	μ_{4}	μ_{7}	μ_{14}
μ_{18}	μ_{18}	μ_{19}	μ_{16}	μ_{25}	μ_{30}	μ_{31}	μ_{20}	μ_{21}	μ_{26}	μ_{27}	μ_{24}	μ_{17}	μ_{22}	μ_{23}	μ_{28}	μ_{29}	μ_{10}	μ_{11}	μ_{8}	μ_{1}	μ_{6}	μ_{7}	μ_{12}	μ_{13}	μ_{2}	μ_{3}	μ_{0}	μ_{9}	μ_{14}	μ_{15}	μ_{4}	μ_{5}
μ_{19}	μ_{19}	μ_{26}	μ_{17}	μ_{16}	μ_{31}	μ_{22}	μ_{29}	μ_{20}	μ_{27}	μ_{18}	μ_{25}	μ_{24}	μ_{23}	μ_{30}	μ_{21}	μ_{28}	μ_{11}	μ_{2}	μ_{9}	μ_{8}	μ_{7}	μ_{14}	μ_{5}	μ_{12}	μ_{3}	μ_{10}	μ_{1}	μ_{0}	μ_{15}	μ_{6}	μ_{13}	μ_{4}
μ_{20}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{16}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{24}	μ_{17}	μ_{18}	μ_{19}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{0}	μ_{9}	μ_{10}	μ_{11}
μ_{21}	μ_{21}	μ_{28}	μ_{23}	μ_{30}	μ_{17}	μ_{16}	μ_{19}	μ_{26}	μ_{29}	μ_{20}	μ_{31}	μ_{22}	μ_{25}	μ_{24}	μ_{27}	μ_{18}	μ_{13}	μ_{4}	μ_{15}	μ_{6}	μ_{9}	μ_{8}	μ_{11}	μ_{2}	μ_{5}	μ_{12}	μ_{7}	μ_{14}	μ_{1}	μ_{0}	μ_{3}	μ_{10}
μ_{22}	μ_{22}	μ_{31}	μ_{28}	μ_{21}	μ_{18}	μ_{27}	μ_{16}	μ_{17}	μ_{30}	μ_{23}	μ_{20}	μ_{29}	μ_{26}	μ_{19}	μ_{24}	μ_{25}	μ_{14}	μ_{7}	μ_{4}	μ_{13}	μ_{10}	μ_{3}	μ_{8}	μ_{9}	μ_{6}	μ_{15}	μ_{12}	μ_{5}	μ_{2}	μ_{11}	μ_{0}	μ_{1}
μ_{23}	μ_{23}	μ_{22}	μ_{29}	μ_{28}	μ_{19}	μ_{18}	μ_{25}	μ_{16}	μ_{31}	μ_{30}	μ_{21}	μ_{20}	μ_{27}	μ_{26}	μ_{17}	μ_{24}	μ_{15}	μ_{14}	μ_{5}	μ_{4}	μ_{11}	μ_{10}	μ_{1}	μ_{8}	μ_{7}	μ_{6}	μ_{13}	μ_{12}	μ_{3}	μ_{2}	μ_{9}	μ_{0}
μ_{24}	μ_{24}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{16}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{0}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}	μ_{4}	μ_{5}	μ_{6}	μ_{7}
μ_{25}	μ_{25}	μ_{24}	μ_{19}	μ_{26}	μ_{21}	μ_{28}	μ_{31}	μ_{22}	μ_{17}	μ_{16}	μ_{27}	μ_{18}	μ_{29}	μ_{20}	μ_{23}	μ_{30}	μ_{1}	μ_{0}	μ_{11}	μ_{2}	μ_{13}	μ_{4}	μ_{7}	μ_{14}	μ_{9}	μ_{8}	μ_{3}	μ_{10}	μ_{5}	μ_{12}	μ_{15}	μ_{6}
μ_{26}	μ_{26}	μ_{27}	μ_{24}	μ_{17}	μ_{22}	μ_{23}	μ_{28}	μ_{29}	μ_{18}	μ_{19}	μ_{16}	μ_{25}	μ_{30}	μ_{31}	μ_{20}	μ_{21}	μ_{2}	μ_{3}	μ_{0}	μ_{9}	μ_{14}	μ_{15}	μ_{4}	μ_{5}	μ_{10}	μ_{11}	μ_{8}	μ_{1}	μ_{6}	μ_{7}	μ_{12}	μ_{13}
μ_{27}	μ_{27}	μ_{18}	μ_{25}	μ_{24}	μ_{23}	μ_{30}	μ_{21}	μ_{28}	μ_{19}	μ_{26}	μ_{17}	μ_{16}	μ_{31}	μ_{22}	μ_{29}	μ_{20}	μ_{3}	μ_{10}	μ_{1}	μ_{0}	μ_{15}	μ_{6}	μ_{13}	μ_{4}	μ_{11}	μ_{2}	μ_{9}	μ_{8}	μ_{7}	μ_{14}	μ_{5}	μ_{12}
μ_{28}	μ_{28}	μ_{29}	μ_{30}	μ_{31}	μ_{24}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	μ_{21}	μ_{22}	μ_{23}	μ_{16}	μ_{25}	μ_{26}	μ_{27}	μ_{4}	μ_{5}	μ_{6}	μ_{7}	μ_{0}	μ_{9}	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{8}	μ_{1}	μ_{2}	μ_{3}
μ_{29}	μ_{29}	μ_{20}	μ_{31}	μ_{22}	μ_{25}	μ_{24}	μ_{27}	μ_{18}	μ_{21}	μ_{28}	μ_{23}	μ_{30}	μ_{17}	μ_{16}	μ_{19}	μ_{26}	μ_{5}	μ_{12}	μ_{7}	μ_{14}	μ_{1}	μ_{0}	μ_{3}	μ_{10}	μ_{13}	μ_{4}	μ_{15}	μ_{6}	μ_{9}	μ_{8}	μ_{11}	μ_{2}
μ_{30}	μ_{30}	μ_{23}	μ_{20}	μ_{29}	μ_{26}	μ_{19}	μ_{24}	μ_{25}	μ_{22}	μ_{31}	μ_{28}	μ_{21}	μ_{18}	μ_{27}	μ_{16}	μ_{17}	μ_{6}	μ_{15}	μ_{12}	μ_{5}	μ_{2}	μ_{11}	μ_{0}	μ_{1}	μ_{14}	μ_{7}	μ_{4}	μ_{13}	μ_{10}	μ_{3}	μ_{8}	μ_{9}
μ_{31}	μ_{31}	μ_{30}	μ_{21}	μ_{20}	μ_{27}	μ_{26}	μ_{17}	μ_{24}	μ_{23}	μ_{22}	μ_{29}	μ_{28}	μ_{19}	μ_{18}	μ_{25}	μ_{16}	μ_{7}	μ_{6}	μ_{13}	μ_{12}	μ_{3}	μ_{2}	μ_{9}	μ_{0}	μ_{15}	μ_{14}	μ_{5}	μ_{4}	μ_{11}	μ_{10}	μ_{1}	μ_{8}

Observations

1. Every element appears only once in every row and every column. Thus every two elements define a third one uniquely i.e $\mathbb{O} \rtimes S^{\circ}$ is a quasigroup.
2. $\mu_0$is the two sided identity of $\mathbb{O} \rtimes s^{\circ}$ i.e $\mu_0 . \mu_i = \mu_i . \mu_0 = \mu_i$, for all $i \in \{0,1, \ldots ,31\}$. We conclude that Octonion split extension forms a loop.
3. $\mathbb{O} \rtimes S^{\circ}$ is not associative i.e $(\mu_i \mu_j)\mu_k \neq \mu_i(\mu_j \mu_k)\ \forall i$

Example 3.3.1

$$\mu_7(\mu_{10}\mu_{14}) = \mu_7 . \mu_{12} = \mu_3 \neq (\mu_7\mu_{10})\mu_{14} = \mu_{13} . \mu_{14} = \mu_{11}$$
$$(e_7, 1)\{(-e_2, 1)(-e_6, 1)\} = (e_7, 1)(e_4, 1) = (e_3, 1)$$
$$\neq \{(e_7, 1)(-e_2, 1)\}(-e_6, 1)$$
$$= (-e_5, 1)(-e_6, 1) = (-e_3, 1)$$

Cyclic subloops of octonion split extensions

The cyclic subloop generated by $\mu_1$is denoted by $\langle \mu_1 \rangle$ and is given by:

$\langle \mu_1 \rangle = \{\mu_1, \mu_8, \mu_9, \mu_0\} = \langle \mu_9 \rangle$

It is constructed as follows:

$\mu_1{}^1 = (e_1, 1)^1 = (e_1, 1) = \mu_1$

$\mu_1{}^2 = (e_1, 1)(e_1, 1) = (-e_0, 1) = \mu_8$

$\mu_1{}^3 = (-e_0, 1)(e_1, 1)=(-e_1, 1) = \mu_9$

$\mu_1{}^4 = (-e_1, 1)(e_1, 1) = (e_0, 1) = \mu_0$

Similarly, other cyclic subloops generated by other elements are:

1. $\langle \mu_2 \rangle = \{\mu_2, \mu_8, \mu_{10}, \mu_0\} = \langle \mu_{10} \rangle$
2. $\langle \mu_3 \rangle = \{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \langle \mu_{11} \rangle$
3. $\langle \mu_4 \rangle = \{\mu_4, \mu_8, \mu_{12}, \mu_0\} = \langle \mu_{12} \rangle$
4. $\langle \mu_5 \rangle = \{\mu_5, \mu_8, \mu_{13}, \mu_0\} = \langle \mu_{13} \rangle$
5. $\langle \mu_6 \rangle = \{\mu_6, \mu_8, \mu_{14}, \mu_0\} = \langle \mu_{14} \rangle$
6. $\langle \mu_7 \rangle = \{\mu_7, \mu_8, \mu_{15}, \mu_0\} = \langle \mu_{15} \rangle$
7. $\langle \mu_8 \rangle = \{\mu_8, \mu_0\}$
8. $\langle \mu_{17} \rangle = \{\mu_{17}, \mu_8, \mu_{25}, \mu_0\} = \langle \mu_{25} \rangle$
9. $\langle \mu_{18} \rangle = \{\mu_{18}, \mu_8, \mu_{26}, \mu_0\} = \langle \mu_{26} \rangle$
10. $\langle \mu_{19} \rangle = \{\mu_{19}, \mu_8, \mu_{27}, \mu_0\} = \langle \mu_{27} \rangle$

Remark 3.3.1.1 In the octonion split extension loop, the commutant is $\langle \mu_8 \rangle = \{\mu_8, \mu_0\}$, and$|\langle \mu_8 \rangle| = 2$.

Theorem 3.3.1 Let $\langle \mu_i \rangle$ be a cyclic subloop of $\mathbb{O} \rtimes S^{\circ}$, then:

1. The elements of the cyclic subloop $\langle \mu_i \rangle$ can be generated using Nim addition as follows:

$\langle \mu_i \rangle = \{\mu_i, \mu_c, \mu_{c \oplus i}, \mu_0\}$

where, $\mu_i$is the generating element, for all $i \in \{0,1, \ldots ,31\}$ and μ_c is the non-identity element in the commutant, $C(\mathbb{O} \rtimes S^{\circ})$.

For example,

$$\langle \mu_{25} \rangle = \{\mu_{25}, \mu_8, \mu_{25 \oplus 8}, \mu_0\} = \{\mu_{25}, \mu_8, \mu_{17}, \mu_0\}$$

2. $\langle \mu_i \rangle = \langle \mu_{i \oplus c} \rangle\ \forall\ \mu_i \in \mathbb{O} \rtimes S^{\circ}$and c the subscript for the non-identity element in $C(\mathbb{O} \rtimes S^{\circ})$ provided that $i \neq 0$.

Coset decomposition

We now construct the distinct cosets using some cyclic subloops of the octonion split extensions. *Examples 3.3.2.1*

1. Let S = $\langle \mu_1 \rangle = \{\mu_1, \mu_8, \mu_9, \mu_0\} = \langle \mu_9 \rangle$ then,

$$\mu_2 S = \mu_2\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{11}, \mu_{10}, \mu_3, \mu_2\},$$
$$\mu_4 S = \mu_4\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{13}, \mu_{12}, \mu_5, \mu_4\},$$
$$\mu_6 S = \mu_6\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_7, \mu_{14}, \mu_{15}, \mu_6\},$$
$$\mu_{16} S = \mu_{16}\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{25}, \mu_{24}, \mu_{17}, \mu_{16}\},$$
$$\mu_{18} S = \mu_{18}\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{19}, \mu_{26}, \mu_{27}, \mu_{18}\},$$
$$\mu_{20} S = \mu_{20}\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{21}, \mu_{28}, \mu_{29}, \mu_{20}\},$$
$$\mu_{22} S = \mu_{22}\{\mu_1, \mu_8, \mu_9, \mu_0\} = \{\mu_{31}, \mu_{30}, \mu_{23}, \mu_{22}\}$$
$$S\mu_2 = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_2 = \{\mu_3, \mu_{10}, \mu_{11}, \mu_2\},$$
$$S\mu_4 = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_4 = \{\mu_5, \mu_{12}, \mu_{13}, \mu_4\},$$
$$S\mu_6 = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_6 = \{\mu_{15}, \mu_{14}, \mu_7, \mu_6\},$$
$$S\mu_{16} = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_{16} = \{\mu_{17}, \mu_{24}, \mu_{25}, \mu_{16}\},$$
$$S\mu_{18} = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_{18} = \{\mu_{27}, \mu_{26}, \mu_{19}, \mu_{18}\},$$
$$S\mu_{20} = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_{20} = \{\mu_{29}, \mu_{28}, \mu_{21}, \mu_{20}\},$$
$$S\mu_{22} = \{\mu_1, \mu_8, \mu_9, \mu_0\}\mu_{22} = \{\mu_{23}, \mu_{30}, \mu_{31}, \mu_{22}\}$$

2. For S= $\langle \mu_3 \rangle = \{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \langle \mu_{11} \rangle$ then,

$$\mu_1 S = \mu_1\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{10}, \mu_9, \mu_2, \mu_1\},$$
$$\mu_4 S = \mu_4\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{15}, \mu_{12}, \mu_7, \mu_4\},$$
$$\mu_5 S = \mu_5\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_6, \mu_{13}, \mu_{14}, \mu_5\},$$
$$\mu_{16} S = \mu_{16}\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{27}, \mu_{24}, \mu_{19}, \mu_{16}\},$$
$$\mu_{17} S = \mu_{17}\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{18}, \mu_{25}, \mu_{26}, \mu_{17}\},$$
$$\mu_{20} S = \mu_{20}\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{23}, \mu_{28}, \mu_{31}, \mu_{20}\},$$
$$\mu_{22} S = \mu_{22}\{\mu_3, \mu_8, \mu_{11}, \mu_0\} = \{\mu_{21}, \mu_{30}, \mu_{29}, \mu_{22}\}$$
$$S\mu_1 = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_1 = \{\mu_2, \mu_9, \mu_{10}, \mu_1\},$$
$$S\mu_4 = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_4 = \{\mu_7, \mu_{12}, \mu_{15}, \mu_4\},$$
$$S\mu_5 = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_5 = \{\mu_{14}, \mu_{13}, \mu_6, \mu_5\},$$
$$S\mu_{16} = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_{16} = \{\mu_{19}, \mu_{24}, \mu_{27}, \mu_{16}\},$$
$$S\mu_{17} = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_{17} = \{\mu_{26}, \mu_{25}, \mu_{18}, \mu_{17}\},$$
$$S\mu_{20} = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_{20} = \{\mu_{31}, \mu_{28}, \mu_{23}, \mu_{20}\},$$
$$S\mu_{22} = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}\mu_{22} = \{\mu_{29}, \mu_{30}, \mu_{21}, \mu_{22}\}$$

In a similar way, the left and right cosets of the other cyclic subloops of the octonion split extensions can be obtained.

Theorem 3.3.2 Let $S = \langle \mu_i \rangle = \{\mu_i, \mu_c, \mu_{(i \oplus c)}, \mu_0\}$, be a cyclic subloop of $\mathbb{O} \rtimes S^{\circ}$, the left cosets of S can be obtained by Nim addition as follows:

$$\mu_j S = \{\mu_{j \oplus i}, \mu_{j \oplus c}, \mu_{j \oplus (i \oplus c)}, \mu_{j \oplus 0}\}$$

where $\mu_j \in \mathbb{O} \rtimes S^\circ$ and $\mu_j \notin S \forall i, j \in \{0, 1, 2, \ldots, 31\}$. On the other hand, the right cosets of S are obtained as follows:

$$S\mu_j = \{\mu_{i \oplus j}, \mu_{c \oplus j}, \mu_{(i \oplus c) \oplus j}, \mu_{0 \oplus j}\}$$

Example 3.3.2.2

If$S = \langle \mu_3 \rangle = \{\mu_3, \mu_8, \mu_{11}, \mu_0\}$, then the subscripts for the elements of $\mu_{12}S$ are obtained by:

$\mu_{12}S \quad = \mu_{12}\{\mu_3, \mu_8, \mu_{11}, \mu_0\}$

$= \{\mu_{12}\mu_3, \mu_{12}\mu_8, \mu_{12}\mu_{11}, \mu_{12}\mu_0\}$

$= \{\mu_{12 \oplus 3}, \mu_{12 \oplus 8}, \mu_{12 \oplus 11}, \mu_{12 \oplus 0}\}$

$= \{\mu_{15}, \mu_4, \mu_7, \mu_{12}\}$.

Conclusions

Given$\forall\ \mu_i, \mu_j \in \mathbb{O} \rtimes S^\circ, i, j \in \{0, 1, 2, \ldots, 31\}$

(i) $\mu_i \in \mu_i S$ or $\mu_i \in S\mu_i$
(ii) $\mu_i S = \mu_j S$ and $\mu_i S \cap \mu_j S = \emptyset$ or $S\mu_i = S\mu_j$ and $S\mu_i \cap S\mu_j = \emptyset$
(iii) $|\mu_i S| = |\mu_j S|$ or $|S\mu_i| = |S\mu_j|$.

Therefore, the left or right cosets of a cyclic subloop$S$ partition $\mathbb{O} \rtimes S^\circ$ into equivalence classes under the relation $\mu_i \sim \mu_j$.

Quotient loops arising from octonion split extensions

If $S = \langle \mu_8 \rangle = \{\mu_8, \mu_0\}$ is a cyclic normal subloop of$\mathbb{O} \rtimes S^\circ$of order 2, then the left cosets of S are:

$\mu_0 S = \mu_0\{\mu_8, \mu_0\} = \{\mu_8, \mu_0\}$, $\mu_1 S = \mu_1\{\mu_8, \mu_0\} = \{\mu_9, \mu_1\}$,

$\mu_2 S = \mu_2\{\mu_8, \mu_0\} = \{\mu_{10}, \mu_2\}$, $\mu_3 S = \mu_3\{\mu_8, \mu_0\} = \{\mu_{11}, \mu_3\}$,

$\mu_4 S = \mu_4\{\mu_8, \mu_0\} = \{\mu_{12}, \mu_4\}$, $\mu_5 S = \mu_5\{\mu_8, \mu_0\} = \{\mu_{13}, \mu_5\}$,

$\mu_6 S = \mu_6\{\mu_8, \mu_0\} = \{\mu_{14}, \mu_6\}$, $\mu_7 S = \mu_7\{\mu_8, \mu_0\} = \{\mu_{15}, \mu_7\}$,

$\mu_{16} S = \mu_{16}\{\mu_8, \mu_0\} = \{\mu_{24}, \mu_{16}\}$, $\mu_{17} S = \mu_{17}\{\mu_8, \mu_0\} = \{\mu_{25}, \mu_{17}\}$,

$\mu_{18} S = \mu_{18}\{\mu_8, \mu_0\} = \{\mu_{26}, \mu_{18}\}$, $\mu_{19} S = \mu_{19}\{\mu_8, \mu_0\} = \{\mu_{27}, \mu_{19}\}$,

$\mu_{20} S = \mu_{20}\{\mu_8, \mu_0\} = \{\mu_{28}, \mu_{20}\}$, $\mu_{21} S = \mu_{21}\{\mu_8, \mu_0\} = \{\mu_{29}, \mu_{21}\}$,

$\mu_{22} S = \mu_{22}\{\mu_8, \mu_0\} = \{\mu_{30}, \mu_{22}\}$, and $\mu_{23} S = \mu_{23}\{\mu_8, \mu_0\} = \{\mu_{31}, \mu_{23}\}$

Therefore,the elements of $\mathbb{O} \rtimes S^\circ / S$ are: $\mu_0 S$, $\mu_1 S$, $\mu_2 S$, $\mu_3 S$, $\mu_4 S$, $\mu_5 S$, $\mu_6 S$, $\mu_7 S$, $\mu_{16} S$, $\mu_{17} S$, $\mu_{18} S$ $\mu_{22} S$, and $\mu_{23} S$. The multiplication of these elements is given by the following table.

Table 5: Multiplication of the elements of $\mathbb{O} \rtimes S^\circ/S$

	μ_0S	μ_1S	μ_2S	μ_3S	μ_4S	μ_5S	μ_6S	μ_7S	$\mu_{16}S$	$\mu_{17}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{22}S$	$\mu_{23}S$
μ_0S	μ_0S	μ_1S	μ_2S	μ_3S	μ_4S	μ_5S	μ_6S	μ_7S	$\mu_{16}S$	$\mu_{17}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{22}S$	$\mu_{23}S$
μ_1S	μ_1S	μ_0S	μ_3S	μ_2S	μ_5S	μ_4S	μ_7S	μ_6S	$\mu_{17}S$	$\mu_{16}S$	$\mu_{19}S$	$\mu_{18}S$	$\mu_{21}S$	$\mu_{20}S$	$\mu_{23}S$	$\mu_{22}S$
μ_2S	μ_2S	μ_3S	μ_0S	μ_1S	μ_6S	μ_7S	μ_4S	μ_5S	$\mu_{18}S$	$\mu_{19}S$	$\mu_{16}S$	$\mu_{17}S$	$\mu_{22}S$	$\mu_{23}S$	$\mu_{20}S$	$\mu_{21}S$
μ_3S	μ_3S	μ_2S	μ_1S	μ_0S	μ_7S	μ_6S	μ_5S	μ_4S	$\mu_{19}S$	$\mu_{18}S$	$\mu_{17}S$	$\mu_{16}S$	$\mu_{23}S$	$\mu_{22}S$	$\mu_{21}S$	$\mu_{20}S$
μ_4S	μ_4S	μ_5S	μ_6S	μ_7S	μ_0S	μ_1S	μ_2S	μ_3S	$\mu_{20}S$	$\mu_{21}S$	$\mu_{22}S$	$\mu_{23}S$	$\mu_{16}S$	$\mu_{17}S$	$\mu_{18}S$	$\mu_{19}S$
μ_5S	μ_5S	μ_4S	μ_7S	μ_6S	μ_1S	μ_0S	μ_3S	μ_2S	$\mu_{21}S$	$\mu_{20}S$	$\mu_{23}S$	$\mu_{22}S$	$\mu_{17}S$	$\mu_{16}S$	$\mu_{19}S$	$\mu_{18}S$
μ_6S	μ_6S	μ_7S	μ_4S	μ_5S	μ_2S	μ_3S	μ_0S	μ_1S	$\mu_{22}S$	$\mu_{23}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{16}S$	$\mu_{17}S$
μ_7S	μ_7S	μ_6S	μ_5S	μ_4S	μ_3S	μ_2S	μ_1S	μ_0S	$\mu_{23}S$	$\mu_{22}S$	$\mu_{21}S$	$\mu_{20}S$	$\mu_{19}S$	$\mu_{18}S$	$\mu_{17}S$	$\mu_{16}S$
$\mu_{16}S$	$\mu_{16}S$	$\mu_{17}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{22}S$	$\mu_{23}S$	μ_0S	μ_1S	μ_2S	μ_3S	μ_4S	μ_5S	μ_6S	μ_7S
$\mu_{17}S$	$\mu_{17}S$	$\mu_{16}S$	$\mu_{19}S$	$\mu_{18}S$	$\mu_{21}S$	$\mu_{20}S$	$\mu_{23}S$	$\mu_{22}S$	μ_1S	μ_0S	μ_3S	μ_2S	μ_5S	μ_4S	μ_7S	μ_6S
$\mu_{18}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{16}S$	$\mu_{17}S$	$\mu_{22}S$	$\mu_{23}S$	$\mu_{20}S$	$\mu_{21}S$	μ_2S	μ_3S	μ_0S	μ_1S	μ_6S	μ_7S	μ_4S	μ_5S
$\mu_{19}S$	$\mu_{19}S$	$\mu_{18}S$	$\mu_{17}S$	$\mu_{16}S$	$\mu_{23}S$	$\mu_{22}S$	$\mu_{21}S$	$\mu_{20}S$	μ_3S	μ_2S	μ_1S	μ_0S	μ_7S	μ_6S	μ_5S	μ_4S
$\mu_{20}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{22}S$	$\mu_{23}S$	$\mu_{16}S$	$\mu_{17}S$	$\mu_{18}S$	$\mu_{19}S$	μ_4S	μ_5S	μ_6S	μ_7S	μ_0S	μ_1S	μ_2S	μ_3S
$\mu_{21}S$	$\mu_{21}S$	$\mu_{20}S$	$\mu_{23}S$	$\mu_{22}S$	$\mu_{17}S$	$\mu_{16}S$	$\mu_{19}S$	$\mu_{18}S$	μ_5S	μ_4S	μ_7S	μ_6S	μ_1S	μ_0S	μ_3S	μ_2S
$\mu_{22}S$	$\mu_{22}S$	$\mu_{23}S$	$\mu_{20}S$	$\mu_{21}S$	$\mu_{18}S$	$\mu_{19}S$	$\mu_{16}S$	$\mu_{17}S$	μ_6S	μ_7S	μ_4S	μ_5S	μ_2S	μ_3S	μ_0S	μ_1S
$\mu_{23}S$	$\mu_{23}S$	$\mu_{22}S$	$\mu_{21}S$	$\mu_{20}S$	$\mu_{19}S$	$\mu_{18}S$	$\mu_{17}S$	$\mu_{16}S$	μ_7S	μ_6S	μ_5S	μ_4S	μ_3S	μ_2S	μ_1S	μ_0S

The Nim addition table of the elements 0, 1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22 and 23 can easily be obtained using the rules for nim addition in the preliminaries section.

Next, we let$M = \langle \mu_1 \rangle = \{\mu_1, \mu_8, \mu_9, \mu_0\} = \langle \mu_9 \rangle$ be a cyclic normal subloop of order 4. The left cosets of M are:

$\mu_0 M = \{\mu_1, \mu_8, \mu_9, \mu_0\}$,$\mu_2 M = \{\mu_{11}, \mu_{10}, \mu_3, \mu_2\}$,

$\mu_4 M = \{\mu_{13}, \mu_{12}, \mu_5, \mu_4\}$,$\mu_6 M = \{\mu_7, \mu_{14}, \mu_{15}, \mu_6\}$,

$\mu_{16} M = \{\mu_{25}, \mu_{24}, \mu_{17}, \mu_{16}\}$,$\mu_{18} M = \{\mu_{19}, \mu_{26}, \mu_{27}, \mu_{18}\}$,

$\mu_{20} M = \{\mu_{21}, \mu_{28}, \mu_{29}, \mu_{20}\}$, and $\mu_{22} M = \{\mu_{31}, \mu_{30}, \mu_{23}, \mu_{22}\}$.

In this case, the elements of $\mathbb{O} \rtimes S^\circ/M$ are, $\mu_0 M$, $\mu_2 M$, $\mu_4 M$, $\mu_6 M$, $\mu_{16} M$, $\mu_{18} M$, $\mu_{20} M$, and $\mu_{22} M$. The multiplication of these elements is given by the following table.

Table 6: Multiplication of the elements of $\mathbb{O} \rtimes S^{\circ}/M$

	$\mu_0 M$	$\mu_2 M$	$\mu_4 M$	$\mu_6 M$	$\mu_{16} M$	$\mu_{18} M$	$\mu_{20} M$	$\mu_{22} M$
$\mu_0 M$	$\mu_0 M$	$\mu_2 M$	$\mu_4 M$	$\mu_6 M$	$\mu_{16} M$	$\mu_{18} M$	$\mu_{20} M$	$\mu_{22} M$
$\mu_2 M$	$\mu_2 M$	$\mu_0 M$	$\mu_6 M$	$\mu_4 M$	$\mu_{18} M$	$\mu_{16} M$	$\mu_{22} M$	$\mu_{20} M$
$\mu_4 M$	$\mu_4 M$	$\mu_6 M$	$\mu_0 M$	$\mu_2 M$	$\mu_{20} M$	$\mu_{22} M$	$\mu_{16} M$	$\mu_{18} M$
$\mu_6 M$	$\mu_6 M$	$\mu_4 M$	$\mu_2 M$	$\mu_0 M$	$\mu_{22} M$	$\mu_{20} M$	$\mu_{18} M$	$\mu_{16} M$
$\mu_{16} M$	$\mu_{16} M$	$\mu_{18} M$	$\mu_{20} M$	$\mu_{22} M$	$\mu_0 M$	$\mu_2 M$	$\mu_4 M$	$\mu_6 M$
$\mu_{18} M$	$\mu_{18} M$	$\mu_{16} M$	$\mu_{22} M$	$\mu_{20} M$	$\mu_2 M$	$\mu_0 M$	$\mu_6 M$	$\mu_4 M$
$\mu_{20} M$	$\mu_{20} M$	$\mu_{22} M$	$\mu_{16} M$	$\mu_{18} M$	$\mu_4 M$	$\mu_6 M$	$\mu_0 M$	$\mu_2 M$
$\mu_{22} M$	$\mu_{22} M$	$\mu_{20} M$	$\mu_{18} M$	$\mu_{16} M$	$\mu_6 M$	$\mu_4 M$	$\mu_2 M$	$\mu_0 M$

To get the Nim addition table for the elements 0, 2, 4, 6, 16, 18, 20 and 22, refer rules for nim addition in the preliminaries section.

Observations

From the above tables we observe that $\forall\, i, j, k \in \{0,1, \ldots ,31\}$ and for S a subloop of $\mathbb{O} \rtimes S^{\circ}$

(i) $\mu_i S.\mu_j S = \mu_{i \oplus j} S$

(ii) $\mu_i S.\mu_j S = \mu_j S.\mu_i S$ since $i \oplus j = j \oplus i$

(iii) $\mu_i S.\mu_i S = \mu_0 S$ since $i \oplus i = 0$

(iv) $\mu_i S(\mu_j S \mu_k S) = (\mu_i S \mu_j S)\mu_k S$ since $i \oplus (j \oplus k) = (i \oplus j) \oplus k$

For a subloop S of order 2 the number of distinct cosets is given by; $\frac{32}{2} = 16$.

We conclude that the subscripts of the elements of the quotient loop generated from octonion split extension are closed under Nim addition. The multiplication of the elements of the quotient loop can thus be achieved by Nim addition.

IV. CONCLUSION

The multiplication tables for the split extension of hypercomplex numbers were constructed by use of the Jonathan-Smith doubling formula. Cyclic subloops were then constructed. The results showed that the cyclic subloops of the split extension of hypercomplex numbers are either of order 2 or 4. Coset decomposition was carried out and the results also showed that the cosets of a cyclic subloop of a split extension loop form a partition of the loop i.e. any two left or right cosets of a cyclic subloop of a split extension loop are either disjoint or identical. Nim addition was also used to give a general way of generating cyclic subloops and distinct cosets arising from them.Finally, in chapter four, quotient loops were constructed and the results confirmed that the multiplication of the elements in the quotient loop is achievable by just considering the Nim-addition of the subscripts of the individual elements.

REFERENCES

[1] Drapal A., Griggs T. S. (2016). On Cosets in Steiner Loops.Buletinul Academiei de Stiinte Number 1(80), Pages, 118-124.

[2] Kimeu A. N. (2008). Some Properties of Hypercomplex Number Basal Elements. Kenyatta University: Nairobi. Kenya.

[3] Kinyon M, Pula K and Vojtechovsky P. (2012). Incidence Properties of Cosets in Loops. J. Combin. Designs 20, 179-197.

[4] Kivunge B. M. (2004). Sedenion extension loops and frames of hypercomplex 2n-ons. Ph. D. Thesis: Iowa State University.

[5] Macharia D. K. (2008). Some Loop Properties of Sedenions. Kenyatta University.

[6] Magero F.B. (2007). Split Extensions of Complex Numbers, Quaternions and Octonion Basis. Kenyatta University: Nairobi, Kenya.

[7] Masila J. M. (2015). The Walsh Functions Obtained from the Frame Multiplication of General Hypercomplex Numbers. Kenyatta University: Nairobi.

[8] Njuguna L. N. (2012). Investigating Sedenion Extension Loops and Frames of Hypercomplex Numbers. Ph. D Thesis: Kenyatta University.

[9] Smith, J. D. H. (1995). A Left Loop on the 15-sphere. Journal of Algebra 176, 135.

[10] Smith, W. D. (2004). Quarternions, Octonions, and now, 16-ons and 2^n-ons; New kinds of numbers. 1.