Mathematically Modeling the Telephone and Telegraph Equations in a Long Cable or Transmission Line

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Abstract: **This paper presents a mathematical model of the telephone and telegraph equations in a transmission line. The flow of electricity in a transmission line was considered. It was observed that waves and wave propagation are ubiquitous in nature and lead to many puzzling questions. For example, why is it that a storm over the ocean sets off a steady swell of small-amplitude waves, but an earthquake at the sea floor can release an enormous flood wave? What produces mirages in the desert? In response to these questions, a great deal of mathematics has been developed to understand and predict the dynamics of light or sound waves the propagation of matter waves, the vibration patterns of elastic bodies, or the peculiar nature of water waves. This paper will provide a guided tour of mathematical wave theory together with physical applications, including the linear wave equation, dispersion and nonlinear waves and solitons. Mathematically one – dimensional heat equations for the flow were derived.**

Keywords: **wave propagation, nonlinear waves and solitons.**

I. INTRODUCTION

This paper described key modeling concept for mobile communication systems operated in outdoor environments. In transmission line there are many interactions between electro-magrictic waves, the antennas which launch and receive them and the environment through which they propagate, in order to understand outdoor mobile propagation. Sometimes these effects are treated using detailed physical models but more usually they are considered too complex and are treated in an empirical or statistical manner. This paper examines how these effects are modeled for telephone and telegraphs which together comprise the main system types used in cellular mobile communications. The transmitted information, encoded as suitable wave forms by the transmitter is modified by channel noise in ways which may be more or less unpredictable to the receiver, so the receiver must be designed to overcome these modifications and hence to deliver the information to its final destination with as few errors or distortions as possible. This is detected by the telephone and telegraph equations. It was noted that waves and wave propagation are ubiquitous in nature and lead to many puzzling questions. For example, why is it that a storm over the ocean sets office steady swell of small-amplitude waves but an earthquake at the sea floor can release an enormous flood wave? Can you hear the shape of a drum? In response to these questions, a great deal of mathematics has been developed to understand and predict the dynamics of light or sound waves, the propagation of matter waves, the vibration of patterns of elastic bodies or the peculiar nature of water waves.

II. A ONE DIMENSIONAL WAVE EQUATION MODEL

We consider the flow of electricity in a long cable (or transmission line). We assume the transmission line to be imperfectly insulated so that there is both capacitance and current leakage to the ground. See figure I below.

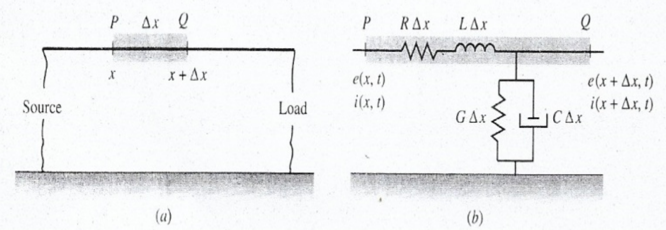

FIGURE 1
A typical element of a transmission line.

We make the following assumptions. Specifically, we let

x = distance from the sending end of the transmission line

e (x,t) = potential at any point on the transmission line at any time.

i (x,t) = current at any point on the transmission line at any time

R = resistance of the transmission line per unit length.

L = inductance of the transmission line per unit length.

G = Conductance to the ground per unit length of the transmission line

C = capacitance to the ground per unit length of transmission line.

Since the potential at Q is equal to the potential at P minus the drop in potential along the element PQ, we see from the equivalent circuit shown in figure 1 (b) that

$$e(x+\Delta x,t) = e(x,t) - (R\Delta x)\, i - (L\Delta x)\frac{\partial i}{\partial t}$$

e (x+Δx,t) - e (x,t) = – (RΔx) i – (LΔx)$\frac{\partial i}{\partial t}$

On, dividing by Δx and then letting Δx tend to zero gives

$$\frac{\partial e}{\partial x} = -Ri - L\frac{\partial i}{\partial t} \qquad (1)$$

Likewise, the current at Q is equal to the current at P minus the current loss through leakage to ground and the apparent current loss due to the varying charge stored on the element. Hence, referring again to figure I (b), we have

i (x+$\Delta x, t$) = i (x, t) – (GΔx) e – (CΔx) $\frac{\partial e}{\partial t}$

or i (x+Δx,t) – i (x, t) = - (GΔx) e – (C Δx)$\frac{\partial e}{\partial t}$ (2)

Dividing byΔx and letting Δx approach zero gives

$$\frac{\partial i}{\partial x} = -Ge - C\frac{\partial e}{\partial t} \qquad (3)$$

If we differentiate equation (1) with respect to x and equation (3) with respect to t, we obtain

$$\frac{\partial^2 e}{\partial x^2} = R\frac{\partial i}{\partial x} - L\frac{\partial^2 i}{\partial x \partial t} \qquad (4)$$

$$\frac{\partial^2 i}{\partial t \partial x} = -G\frac{\partial e}{\partial t} - C\frac{\partial^2 e}{\partial t^2} \qquad (5)$$

We eliminate the term $\frac{\partial^2 i}{\partial t \partial x} \equiv \frac{\partial^2 i}{\partial x \partial t}$between these two equations and then substitute for $\frac{\partial i}{\partial x}$ from (3), to obtain

$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} + (RC + GL)\frac{\partial e}{\partial t} + RGe \qquad (6)$$

By differentiating (1) with respect to t and equation (3) with respect to x and then eliminating the derivatives of e, we obtain a similar equation for i.e$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + RGi$ (7)

Equations (6 and (7) are known as the telephone equations. They assert that e and i satisfy the same partial differential equation. Two special cases of the telephone equations are worthy of note:

Case (i) if leakage and inductance are negligible ie. if G = L = O, as they are, for example, for coaxial cables, Equations (6) and (7) reduce, respectively, to

$$\frac{\partial^2 e}{\partial x^2} = RC\frac{\partial e}{\partial t} \qquad (8)$$

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \qquad (9)$$

These are known as the telegraphy equations

Mathematically, they are identical with the one-dimensional heat equation

$$a^2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad a^2 = \frac{C\ell}{Kg} \qquad (10)$$

case (ii) At high frequencies the factor introduced by the time differentiation is large. Hence the terms involving e and $\frac{\partial e}{\partial t}$ or i and $\frac{\partial i}{\partial t}$ are insignificant in comparison with the terms containing the corresponding second derivatives $\frac{\partial^2 e}{\partial x^2}$and $\frac{\partial^2 i}{\partial t^2}$

In this case Equations (6) and (7) reduce respectively to

$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \qquad (11)$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \qquad (12)$$

Each of these is an example of the one - dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}; a^2 = \frac{Tg}{w}$$

$\frac{1}{\sqrt{LC}}$having, in fact the dimensions of velocity. These equations are obtained at any frequency of course if R = G = O

III. RESULT AND CONCLUSION

It is interesting to note that nowhere in the derivation of any of the preceding equations was any use made of boundary conditions. In other words, the same partial differential equation is satisfied by the deflections of a membrane whether the membrane is round or square, the same equation is satisfied by the deflections of a vibrating beam whether the beam is a cantilever, fixed at one end and free at the other, or a highway bridge, held in place at both ends. Likewise, the flow of heat in a rod is described by the same equation whether the ends of the rod are maintained at constant temperatures, insulated, or allowed to cool by radiation into the surrounding air, (Ray Wylie & Louis, 2008).

Of course had we chosen to use polar rather than rectangular coordinates to study the vibrations of a membrane or the flow of heat in a thin metal sheet, we would have obtained different equations, but again the derivations would not have been influenced by any boundary conditions.

The Benard problem is one of two well-known problems of wave dynamics in which R and G are real. The other one is the Taylor problem of couette flow between rotating cylinders. In most other problems R is complex, and the marginal state (R = G = O) contains propagating waves. In the Benard and Taylor problem, however, the marginal state corresponds to R = O, G = O, and is therefore stationary and does not contain propagating waves. In these the onset of instability is marked by a transmission from the background state to another steady state. In such a case it is commonly said that the principle of exchange of stabilities is valid, and the instability sets in as a cellular convection.

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