

# Primitive of First Kind Volterra Integro-Differential Equation Using Shehu Transform

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**Abstract:** Volterra integro-differential equation generally appears when an initial value problem is to be converted into an integral equation. In this paper, authors determined the primitive of first kind Volterra integro-differential equation using Shehu transform. In this work, authors have considered that the kernel of first kind Volterra integro-differential equation is a convolution type kernel. Some numerical problems have been considered and solved with the help of Shehu transform for explaining the complete methodology. Results of numerical problems show that Shehu transform is very effective integral transform for determining the primitive of first kind Volterra integro-differential equation.

**Keywords:** Volterra integro-differential equation; Shehu transform; Convolution; Inverse Shehu transform.

## I. INTRODUCTION

Integral transforms are widely used mathematical techniques because these techniques provide the exact solution of the problem without large calculation work. Nowadays, these mathematical techniques are rapidly used for solving the problems of applied mathematics, theoretical mechanics, statistics, mathematical physics and pharmacokinetics. Aggarwal and other scholars [1-8] used different integral transformations (Mahgoub, Aboodh, Shehu, Elzaki, Mohand, Kamal) and determined the analytical solutions of first and second kind Volterra integral equations. Solutions of the problems of Volterra integro-differential equations of second kind are given by Aggarwal et al. [9-11] with the help of Mahgoub, Kamal and Aboodh transformations. In the year 2018, Aggarwal with other scholars [12-13] determined the solutions of linear partial integro-differential equations using Mahgoub and Kamal transformations. Aggarwal et al. [14-20] used Sawi; Mohand; Kamal; Shehu; Elzaki; Laplace and Mahgoub transformations and determined the solutions of population growth and decay problems by the help of their mathematical models. Aggarwal et al. [21-26] defined dualities relations of many advance integral transformations. Comparative studies of Mohand and other integral transformations are given by Aggarwal et al. [27-31]. Aggarwal et al. [32-39] defined Elzaki; Aboodh; Shehu; Sumudu; Mohand; Kamal; Mahgoub and Laplace transformations of error function with applications. The solutions of ordinary differential equations with variable coefficients are given by Aggarwal et al. [40] using Mahgoub

transform. Aggarwal et al. [41-45] used different integral transformations and determined the solutions of Abel's integral equations. Aggarwal et al. [46-49] worked on Bessel's functions and determined their Mohand; Aboodh; Mahgoub and Elzaki transformations.

Chaudhary et al. [50] gave the connections between Aboodh transform and some useful integral transforms. Aggarwal et al. [51-52] used Elzaki and Kamal transforms for solving linear Volterra integral equations of first kind. Solution of population growth and decay problems was given by Aggarwal et al. [53-54] by using Aboodh and Sadik transformations respectively. Aggarwal and Sharma [55] defined Sadik transform of error function. Application of Sadik transform for handling linear Volterra integro-differential equations of second kind was given by Aggarwal et al. [56]. Aggarwal and Bhatnagar [57] gave the solution of Abel's integral equation using Sadik transform. A comparative study of Mohand and Mahgoub transforms was given by Aggarwal [58]. Aggarwal [59] defined Kamal transform of Bessel's functions. Chauhan and Aggarwal [60] used Laplace transform and solved convolution type linear Volterra integral equation of second kind. Sharma and Aggarwal [61] applied Laplace transform and determined the solution of Abel's integral equation. Laplace transform for the solution of first kind linear Volterra integral equation was given by Aggarwal and Sharma [62]. Mishra et al. [63] defined the relationship between Sumudu and some efficient integral transforms.

The main aim of this paper is to determine the primitive of first kind Volterra integro-differential equation using Shehu transform.

## II. DEFINITION OF SHEHU TRANSFORM

The Shehu transform of the function  $G(t)$  for all  $t \geq 0$  is defined as [64]

$$S\{G(t)\} = \int_0^{\infty} G(t)e^{-\frac{vt}{u}} dt = H(v, u), v > 0, u > 0,$$

where operator  $S$  is called the Shehu transform operator.

*Fundamental Properties of Shehu Transform*

*Linearity property* [4]:  $\left[ \begin{array}{l} \text{If } S\{G_1(t)\} = H_1(v, u) \\ \text{and } S\{G_2(t)\} = H_2(v, u) \end{array} \right]$   
 then  $\left[ \begin{array}{l} S\{aG_1(t) + bG_2(t)\} \\ = aH_1(v, u) + bH_2(v, u) \end{array} \right]$ , where  $a, b$  are arbitrary constants.

*Proof:* By the definition of Shehu transform, we have

$$\begin{aligned} S\{G(t)\} &= \int_0^\infty G(t)e^{-\frac{vt}{u}} dt \\ &\Rightarrow \left[ \begin{array}{l} S\{aG_1(t) + bG_2(t)\} \\ = \int_0^\infty [aG_1(t) + bG_2(t)]e^{-\frac{vt}{u}} dt \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{l} S\{aG_1(t) + bG_2(t)\} \\ = a \int_0^\infty G_1(t)e^{-\frac{vt}{u}} dt + b \int_0^\infty G_2(t)e^{-\frac{vt}{u}} dt \end{array} \right] \\ &\Rightarrow S\{aG_1(t) + bG_2(t)\} = aS\{G_1(t)\} + bS\{G_2(t)\} \\ &\Rightarrow S\{aG_1(t) + bG_2(t)\} = aH_1(v, u) + bH_2(v, u), \end{aligned}$$

where  $a, b$  are arbitrary constants.

*Change of scale property* [34]: If Shehu transform of function  $G(t)$  is  $H(v, u)$  then Shehu transform of function  $G(at)$  is given by  $\frac{1}{a}H\left(\frac{v}{a}, u\right)$ .

*Proof:* By the definition of Shehu transform, we have

$$\begin{aligned} S\{G(at)\} &= \int_0^\infty G(at)e^{-\frac{vt}{u}} dt \\ \text{Put } at &= p \Rightarrow adt = dp \text{ in above equation, we have} \\ S\{G(at)\} &= \frac{1}{a} \int_0^\infty G(p)e^{-\frac{vp}{ua}} dp \\ &\Rightarrow S\{G(at)\} = \frac{1}{a} \left[ \int_0^\infty G(p)e^{-\frac{(v/a)p}{u}} dp \right] \\ &\Rightarrow S\{G(at)\} = \frac{1}{a} H\left(\frac{v}{a}, u\right) \end{aligned}$$

*Shifting property* [34]: If Shehu transform of function  $G(t)$  is  $H(v, u)$  then Shehu transform of function  $e^{at}G(t)$  is given by  $H(v - au, u)$ .

*Proof:* By the definition of Shehu transform, we have

$$\begin{aligned} S\{e^{at}G(t)\} &= \int_0^\infty e^{at}G(t)e^{-\frac{vt}{u}} dt \\ &\Rightarrow S\{e^{at}G(t)\} = \int_0^\infty G(t)e^{-\left(\frac{v}{u}-a\right)t} dt \\ &= \int_0^\infty G(t)e^{-\left(\frac{v-au}{u}\right)t} dt = H(v - au, u) \end{aligned}$$

*Shehu transform of the derivatives of the function  $G(t)$*  [4, 17, 34, 64]:

If  $S\{G(t)\} = H(v, u)$  then

$$\begin{aligned} \text{a)} \quad S\{G'(t)\} &= \frac{v}{u}H(v, u) - G(0) \\ \text{b)} \quad S\{G''(t)\} &= \frac{v^2}{u^2}H(v, u) - \frac{v}{u}G(0) - G'(0) \\ \text{c)} \quad \left[ \begin{array}{l} S\{G^{(n)}(t)\} \\ = \frac{v^n}{u^n}H(v, u) - \sum_{k=0}^{n-1} \left(\frac{v}{u}\right)^{n-(k+1)} G^{(k)}(0) \end{array} \right] \end{aligned}$$

*Shehu transform of integral of a function  $G(t)$*  [34]:

$$\text{If } S\{G(t)\} = H(v, u) \text{ then } \left[ \begin{array}{l} S\left\{\int_0^t G(t)dt\right\} \\ = \frac{u}{v}H(v, u) \end{array} \right].$$

*Proof:* Let  $J(t) = \int_0^t G(t)dt$ . Then  $J'(t) = G(t)$  and  $J(0) = 0$ .

Now by the property of Shehu transform of the derivative of function, we have

$$\begin{aligned} S\{J'(t)\} &= \frac{v}{u}S\{J(t)\} - J(0) = \frac{v}{u}S\{J(t)\} \\ \Rightarrow S\{J(t)\} &= \frac{u}{v}S\{J'(t)\} = \frac{u}{v}S\{G(t)\} \\ \Rightarrow S\{J(t)\} &= \frac{u}{v}H(v, u) \\ \Rightarrow S\left\{\int_0^t G(t)dt\right\} &= \frac{u}{v}H(v, u) \end{aligned}$$

*Convolution theorem for Shehu transforms* [4, 34]: If Shehu transform of functions  $G_1(t)$  and  $G_2(t)$  are  $H_1(v, u)$  and  $H_2(v, u)$  respectively then Shehu transform of their convolution  $G_1(t) * G_2(t)$  is given by

$$\begin{aligned} S\{G_1(t) * G_2(t)\} &= S\{G_1(t)\}S\{G_2(t)\} \\ \Rightarrow S\{G_1(t) * G_2(t)\} &= H_1(v, u)H_2(v, u), \end{aligned}$$

where  $G_1(t) * G_2(t)$  is defined by

$$\left[ \begin{array}{l} G_1(t) * G_2(t) = \int_0^t G_1(t-x)G_2(x)dx \\ = \int_0^t G_1(x)G_2(t-x)dx \end{array} \right].$$

*Proof:* By the definition of Shehu transform, we have

$$\begin{aligned} S\{G_1(t) * G_2(t)\} &= \int_0^\infty e^{-\frac{vt}{u}} [G_1(t) * G_2(t)]dt \\ &\Rightarrow \left[ \begin{array}{l} S\{G_1(t) * G_2(t)\} \\ = \int_0^\infty e^{-\frac{vt}{u}} \left[ \int_0^t G_1(t-x)G_2(x)dx \right] dt \end{array} \right] \end{aligned}$$

By changing the order of integration, we have

$$\left[ \begin{array}{l} S\{G_1(t) * G_2(t)\} \\ = \int_0^\infty G_2(x) \left[ \int_x^\infty e^{-\frac{vt}{u}} G_1(t-x) dt \right] dx \end{array} \right]$$

Put  $t - x = p$  so that  $dt = dp$  in above equation, we have

$$\begin{aligned} & \left[ = \int_0^\infty G_2(x) \left[ \int_0^\infty e^{-\frac{v(p+x)}{u}} G_1(p) dp \right] dx \right] \\ \Rightarrow & \left[ = \int_0^\infty G_2(x) e^{-\frac{xv}{u}} \left[ \int_0^\infty e^{-\frac{pv}{u}} G_1(p) dp \right] dx \right] \\ \Rightarrow & S\{G_1(t) * G_2(t)\} = H_1(v, u)H_2(v, u). \end{aligned}$$

Table 1 Shehu Transform of Frequently Encountered Functions [4, 17, 34, 64]

S.N.	$G(t)$	$S\{G(t)\} = H(v, u)$
1.	1	$\frac{u}{v}$
2.	$t$	$\left(\frac{u}{v}\right)^2$
3.	$t^2$	$2! \left(\frac{u}{v}\right)^3$
4.	$t^n, n \in N$	$n! \left(\frac{u}{v}\right)^{n+1}$
5.	$t^n, n > -1$	$\Gamma(n+1) \left(\frac{u}{v}\right)^{n+1}$
6.	$e^{at}$	$\frac{u}{v-au}$
7.	$\sin at$	$\frac{au^2}{(v^2+a^2u^2)}$
8.	$\cos at$	$\frac{uv}{(v^2+a^2u^2)}$
9.	$\sin hat$	$\frac{au^2}{(v^2-a^2u^2)}$
10.	$\cos hat$	$\frac{uv}{(v^2-a^2u^2)}$
11.	$J_0(at)$	$\frac{u}{\sqrt{(v^2+a^2u^2)}}$
12.	$t \sin at$	$\frac{2avu^3}{(v^2+a^2u^2)^2}$
13.	$t \cos at$	$\frac{u^2(v^2-a^2u^2)}{(v^2+a^2u^2)^2}$

### III. INVERSE SHEHU TRANSFORM [4]

If  $S\{G(t)\} = H(v, u)$  then  $G(t)$  is called the inverse Shehu transform of  $H(v, u)$  and mathematically it is defined as  $G(t) = S^{-1}\{H(v, u)\}$ , where  $S^{-1}$  is the inverse Shehu transform operator.

Linearity property of inverse Shehu transforms [17]:

If  $S^{-1}\{H_1(v, u)\} = G_1(t)$  and  $S^{-1}\{H_2(v, u)\} = G_2(t)$  then

$$\left[ \begin{aligned} & S^{-1}\{a H_1(v, u) + b H_2(v, u)\} \\ & = a S^{-1}\{H_1(v, u)\} + b S^{-1}\{H_2(v, u)\} \end{aligned} \right]$$

$$\Rightarrow \left[ \begin{aligned} & S^{-1}\{a H_1(v, u) + b H_2(v, u)\} \\ & = a G_1(t) + b G_2(t) \end{aligned} \right], \text{ where } a, b \text{ are arbitrary constants.}$$

Table 2 Inverse Shehu Transform of Frequently Encountered Functions [17]

S.N.	$H(v, u)$	$G(t) = S^{-1}\{H(v, u)\}$
1.	$\frac{u}{v}$	1
2.	$\left(\frac{u}{v}\right)^2$	$t$
3.	$\left(\frac{u}{v}\right)^3$	$\frac{t^2}{2!}$
4.	$\left(\frac{u}{v}\right)^{n+1}, n \in N$	$\frac{t^n}{n!}$
5.	$\left(\frac{u}{v}\right)^{n+1}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{u}{v-au}$	$e^{at}$
7.	$\frac{u^2}{(v^2+a^2u^2)}$	$\frac{\sin at}{a}$
8.	$\frac{uv}{(v^2+a^2u^2)}$	$\cos at$
9.	$\frac{u^2}{(v^2-a^2u^2)}$	$\frac{\sin hat}{a}$
10.	$\frac{uv}{(v^2-a^2u^2)}$	$\cos hat$
11.	$\frac{u}{\sqrt{(v^2+a^2u^2)}}$	$J_0(at)$
12.	$\frac{vu^3}{(v^2+a^2u^2)^2}$	$\frac{t \sin at}{2a}$
13.	$\frac{u^2(v^2-a^2u^2)}{(v^2+a^2u^2)^2}$	$t \cos at$

### IV. PRIMITIVE OF FIRST KIND VOLTERRA INTEGRO-DIFFERENTIAL EQUATION USING SHEHU TRANSFORM

In this part of the paper, authors determine the primitive of first kind Volterra integro-differential equation using Shehu transform. In this work, we have considered that the kernel of first kind Volterra integro-differential equation is a convolution type kernel.

Convolution type first kind Volterra integro-differential equation is given by

$$\left. \begin{aligned} & \int_0^t K_1(t-u) \omega(u) du + \\ & \int_0^t K_2(t-u) \omega^{(n)}(u) du \end{aligned} \right\} = F(t), K_2(t-u) \neq 0 \tag{1}$$

$$\text{with } \left. \begin{aligned} & \omega(0) = \delta_0, \omega'(0) = \delta_1, \\ & \omega''(0) = \delta_2, \dots, \dots, \\ & \omega^{(n-1)}(0) = \delta_{n-1} \end{aligned} \right\} \tag{2}$$

$$\left. \begin{aligned} & \left[ \begin{array}{l} K_1(t-u), K_2(t-u) \\ = \text{faltung type kernels} \\ \text{of integral equation} \end{array} \right] \\ & \left[ \begin{array}{l} \omega(t) = \text{unknown} \\ \text{function} \end{array} \right] \\ \text{where } & \left[ \begin{array}{l} \omega^{(n)}(t) = \text{nth derivative} \\ \text{of unknown function} \end{array} \right] \\ & \left[ \begin{array}{l} F(t) = \text{known} \\ \text{function} \end{array} \right] \\ & \left[ \begin{array}{l} \delta_0, \delta_1, \delta_2, \dots, \delta_{n-1} \\ = \text{real numbers} \end{array} \right] \end{aligned} \right\}$$

Taking Shehu transform of both sides of (1), we have

$$\left[ \begin{array}{l} S \left\{ \int_0^t K_1(t-u) \omega(u) du \right\} \\ + S \left\{ \int_0^t K_2(t-u) \omega^{(n)}(u) du \right\} \\ = S\{F(t)\} \end{array} \right] \quad (3)$$

Applying convolution theorem of Shehu transform on (3), we have

$$\left[ \begin{array}{l} S\{K_1(t)\}S\{\omega(t)\} \\ + S\{K_2(t)\}S\{\omega^{(n)}(t)\} = S\{F(t)\} \end{array} \right] \quad (4)$$

Applying the property ‘‘Shehu transform of derivative of functions’’ on (4), we get

$$\left[ \begin{array}{l} S\{K_1(t)\}S\{\omega(t)\} \\ + S\{K_2(t)\} \left[ \begin{array}{l} \frac{v^n}{u^n} S\{\omega(t)\} \\ - \frac{v^{n-1}}{u^{n-1}} \omega(0) \\ - \frac{v^{n-2}}{u^{n-2}} \omega'(0) \\ - \frac{v^{n-3}}{u^{n-3}} \omega''(0) \\ \dots \dots \\ - \omega^{(n-1)}(0) \end{array} \right] \end{array} \right] = S\{F(t)\} \quad (5)$$

Now using (2) in (5), we have

$$\left[ \begin{array}{l} S\{K_1(t)\}S\{\omega(t)\} \\ + S\{K_2(t)\} \left[ \begin{array}{l} \frac{v^n}{u^n} S\{\omega(t)\} \\ - \frac{v^{n-1}}{u^{n-1}} \delta_0 \\ - \frac{v^{n-2}}{u^{n-2}} \delta_1 \\ - \frac{v^{n-3}}{u^{n-3}} \delta_2 \\ \dots \dots \\ - \delta_{n-1} \end{array} \right] \end{array} \right] = S\{F(t)\}$$

$$\Rightarrow \left[ \begin{array}{l} \left[ \begin{array}{l} S\{K_1(t)\} \\ + \frac{v^n}{u^n} S\{K_2(t)\} \end{array} \right] S\{\omega(t)\} \\ S\{F(t)\} \\ + S\{K_2(t)\} \left( \begin{array}{l} v^{n-1} \\ \frac{v^{n-1}}{u^{n-1}} \delta_0 \\ + \frac{v^{n-2}}{u^{n-2}} \delta_1 \\ + \frac{v^{n-3}}{u^{n-3}} \delta_2 \\ + \dots \dots \\ + \delta_{n-1} \end{array} \right) \end{array} \right]$$

$$\Rightarrow S\{\omega(t)\} = \frac{S\{F(t)\} \left( \begin{array}{l} v^{n-1} \\ \frac{v^{n-1}}{u^{n-1}} \delta_0 \\ + \frac{v^{n-2}}{u^{n-2}} \delta_1 \\ + \frac{v^{n-3}}{u^{n-3}} \delta_2 \\ + \dots \dots \\ + \delta_{n-1} \end{array} \right)}{\left[ \begin{array}{l} S\{K_1(t)\} \\ + \frac{v^n}{u^n} S\{K_2(t)\} \end{array} \right]} \neq 0 \quad (6)$$

The inverse Shehu transform of both sides of (6) gives the required solution of given convolution type first kind Volterra integro-differential equation.

### V. NUMERICAL PROBLEMS

In this part of the paper, some numerical problems have been considered for explaining the complete methodology.

*Problem: 1* Consider the following first kind Volterra integro-differential equation

$$\left[ \begin{array}{l} \int_0^t (t-u) \omega(u) du \\ + \int_0^t (t-u)^2 \omega'(u) du \\ = 3t - 3sint \end{array} \right] \quad (7)$$

with  $\omega(0) = 0$  (8)

Taking Shehu transform of both sides of (7), we have

$$\left[ \begin{array}{l} S \left\{ \int_0^t (t-u) \omega(u) du \right\} \\ + S \left\{ \int_0^t (t-u)^2 \omega'(u) du \right\} \\ = S\{3t - 3sint\} \end{array} \right] \quad (9)$$

Applying convolution theorem of Shehu transform on (9), we have

$$\left[ \begin{array}{l} S\{t\}S\{\omega(t)\} + S\{t^2\}S\{\omega'(t)\} \\ = S\{3t - 3sint\} \\ = 3S\{t\} - 3S\{sint\} \end{array} \right]$$

$$\Rightarrow \left[ \begin{aligned} &\left(\frac{u}{v}\right)^2 S\{\omega(t)\} + 2\left(\frac{u}{v}\right)^3 S\{\omega'(t)\} \\ &= 3\left(\frac{u}{v}\right)^2 - \frac{3u^2}{(v^2+u^2)} \end{aligned} \right] \quad (10)$$

Applying the property “Shehu transform of derivative of functions” on (10), we get

$$\left[ \begin{aligned} &\left(\frac{u}{v}\right)^2 S\{\omega(t)\} \\ &+ 2\left(\frac{u}{v}\right)^3 \left[ \frac{v}{u} S\{\omega(t)\} - \omega(0) \right] \\ &= 3\left(\frac{u}{v}\right)^2 - \frac{3u^2}{(v^2+u^2)} \end{aligned} \right] \quad (11)$$

Now using (8) in (11), we have

$$\left[ \begin{aligned} &3\left(\frac{u}{v}\right)^2 S\{\omega(t)\} = 3\left(\frac{u}{v}\right)^2 - \frac{3u^2}{(v^2+u^2)} \\ \Rightarrow &\left[ = 1 - \frac{v^2}{(v^2+u^2)} = \frac{u^2}{(v^2+u^2)} \right] \end{aligned} \right] \quad (12)$$

Taking inverse Shehu transform of both sides of (12), we get the required solution of (7) with (8) as

$$\left[ \omega(t) = S^{-1} \left\{ \frac{u^2}{(v^2+u^2)} \right\} = sint \right].$$

**Problem: 2** Consider the following first kind Volterra integro-differential equation

$$\left[ \begin{aligned} &\int_0^t \sin(t-u) \omega(u) du \\ &- \frac{1}{2} \int_0^t (t-u) \omega''(u) du \\ &= \frac{t}{2} - \frac{tcost}{2} \end{aligned} \right] \quad (13)$$

$$\text{with } [\omega(0) = 0, \omega'(0) = 1] \quad (14)$$

Taking Shehu transform of both sides of (13), we have

$$\left[ \begin{aligned} &S \left\{ \int_0^t \sin(t-u) \omega(u) du \right\} \\ &- \frac{1}{2} S \left\{ \int_0^t (t-u) \omega''(u) du \right\} \\ &= S \left\{ \frac{t}{2} - \frac{tcost}{2} \right\} \end{aligned} \right] \quad (15)$$

Applying convolution theorem of Shehu transform on (15), we have

$$\left[ \begin{aligned} &S\{sint\}S\{\omega(t)\} \\ &- \frac{1}{2} S\{t\}S\{\omega''(t)\} \\ &= \frac{1}{2} S\{t\} - \frac{1}{2} S\{tcost\} \end{aligned} \right] \\ \Rightarrow \left[ \begin{aligned} &\frac{u^2}{(v^2+u^2)} S\{\omega(t)\} \\ &- \frac{1}{2} \left(\frac{u}{v}\right)^2 S\{\omega''(t)\} \\ &= \frac{1}{2} \left(\frac{u}{v}\right)^2 - \frac{1}{2} \left\{ \frac{u^2(v^2-u^2)}{(v^2+u^2)^2} \right\} \end{aligned} \right] \quad (16)$$

Applying the property “Shehu transform of derivative of functions” on (16), we get

$$\left[ \begin{aligned} &\frac{u^2}{(v^2+u^2)} S\{\omega(t)\} \\ &- \frac{1}{2} \left(\frac{u}{v}\right)^2 \left[ \begin{aligned} &\left(\frac{v}{u}\right)^2 S\{\omega(t)\} \\ &- \frac{v}{u} \omega(0) \\ &- \omega'(0) \end{aligned} \right] \\ &= \frac{1}{2} \left(\frac{u}{v}\right)^2 - \frac{1}{2} \left\{ \frac{u^2(v^2-u^2)}{(v^2+u^2)^2} \right\} \end{aligned} \right] \quad (17)$$

Now using (14) in (17), we have

$$\left[ \begin{aligned} &\frac{u^2}{(v^2+u^2)} S\{\omega(t)\} \\ &- \frac{1}{2} \left(\frac{u}{v}\right)^2 \left[ \left(\frac{v}{u}\right)^2 S\{\omega(t)\} - 1 \right] \\ &= \frac{1}{2} \left(\frac{u}{v}\right)^2 - \frac{1}{2} \left\{ \frac{u^2(v^2-u^2)}{(v^2+u^2)^2} \right\} \end{aligned} \right] \\ \Rightarrow S\{\omega(t)\} = \frac{u^2}{(v^2+u^2)} \quad (18)$$

Taking inverse Shehu transform of both sides of (18), we get the required solution of (13) with (14) as

$$\left[ \omega(t) = S^{-1} \left\{ \frac{u^2}{(v^2+u^2)} \right\} = sint \right].$$

**Problem: 3** Consider the following first kind Volterra integro-differential equation

$$\left[ \begin{aligned} &\int_0^t \cos(t-u) \omega(u) du \\ &+ \int_0^t \sin(t-u) \omega'''(u) du \\ &= 1 + sint - cost \end{aligned} \right] \quad (19)$$

$$\text{with } \left[ \begin{aligned} &\omega(0) = 1, \\ &\omega'(0) = 1, \omega''(0) = -1 \end{aligned} \right] \quad (20)$$

Taking Shehu transform of both sides of (19), we have

$$\left[ \begin{aligned} &S \left\{ \int_0^t \cos(t-u) \omega(u) du \right\} \\ &+ S \left\{ \int_0^t \sin(t-u) \omega'''(u) du \right\} \\ &= S\{1 + sint - cost\} \end{aligned} \right] \quad (21)$$

Applying convolution theorem of Shehu transform on (21), we have

$$\left[ \begin{aligned} &S\{cost\}S\{\omega(t)\} \\ &+ S\{sint\}S\{\omega'''(t)\} \\ &= S\{1\} + S\{sint\} - S\{cost\} \end{aligned} \right] \\ \Rightarrow \left[ \begin{aligned} &\left(\frac{uv}{(v^2+u^2)}\right) S\{\omega(t)\} \\ &+ \left(\frac{u^2}{(v^2+u^2)}\right) S\{\omega'''(t)\} \\ &= \frac{u}{v} + \left(\frac{u^2}{(v^2+u^2)}\right) - \left(\frac{uv}{(v^2+u^2)}\right) \end{aligned} \right] \quad (22)$$

Applying the property “Shehu transform of derivative of functions” on (22), we get

$$\begin{aligned} & \left[ \begin{array}{l} \left(\frac{uv}{v^2+u^2}\right) S\{\omega(t)\} \\ + \left(\frac{u^2}{v^2+u^2}\right) \left[ \begin{array}{l} \left(\frac{v}{u}\right)^3 S\{\omega(t)\} \\ - \left(\frac{v}{u}\right)^2 \omega(0) \\ - \left(\frac{v}{u}\right) \omega'(0) \\ - \omega''(0) \end{array} \right] \end{array} \right] \\ & = \frac{u}{v} + \left(\frac{u^2}{v^2+u^2}\right) - \left(\frac{uv}{v^2+u^2}\right) \end{aligned} \quad (23)$$

Now using (20) in (23), we have

$$\begin{aligned} & \left[ \begin{array}{l} \left(\frac{uv}{v^2+u^2}\right) S\{\omega(t)\} \\ + \left(\frac{u^2}{v^2+u^2}\right) \left[ \left(\frac{v}{u}\right)^3 S\{\omega(t)\} - \left(\frac{v}{u}\right)^2 - \left(\frac{v}{u}\right) + 1 \right] \\ = \frac{u}{v} + \left(\frac{u^2}{v^2+u^2}\right) - \left(\frac{uv}{v^2+u^2}\right) \end{array} \right] \\ & \Rightarrow \left[ S\{\omega(t)\} = \left(\frac{u}{v}\right)^2 + \left(\frac{uv}{v^2+u^2}\right) \right] \end{aligned} \quad (24)$$

Taking inverse Shehu transform of both sides of (24), we get the required solution of (19) with (20) as

$$\begin{aligned} & \left[ \begin{array}{l} \omega(t) = S^{-1} \left\{ \left(\frac{u}{v}\right)^2 + \frac{uv}{v^2+u^2} \right\} \\ = S^{-1} \left\{ \left(\frac{u}{v}\right)^2 \right\} + S^{-1} \left\{ \frac{uv}{v^2+u^2} \right\} \end{array} \right] \\ & \Rightarrow \omega(t) = t + cost. \end{aligned}$$

**Problem: 4** consider the following first kind Volterra integro-differential equation

$$\begin{aligned} & \left[ \begin{array}{l} \int_0^t (t-u)^2 \omega(u) du \\ - \frac{1}{12} \int_0^t (t-u)^3 \omega'''(u) du = \frac{t^4}{12} \end{array} \right] \end{aligned} \quad (25)$$

$$\text{with } \left[ \begin{array}{l} \omega(0) = 0, \\ \omega'(0) = 3, \omega''(0) = 0 \end{array} \right] \quad (26)$$

Taking Shehu transform of both sides of (25), we have

$$\begin{aligned} & \left[ \begin{array}{l} S \left\{ \int_0^t (t-u)^2 \omega(u) du \right\} \\ - \frac{1}{12} S \left\{ \int_0^t (t-u)^3 \omega'''(u) du \right\} \\ = \frac{1}{12} S \{t^4\} \end{array} \right] \end{aligned} \quad (27)$$

Applying convolution theorem of Shehu transform on (27), we have

$$\begin{aligned} & \left[ \begin{array}{l} S\{t^2\} S\{\omega(t)\} \\ - \frac{1}{12} S\{t^3\} S\{\omega'''(t)\} = 2 \left(\frac{u}{v}\right)^5 \end{array} \right] \\ & \Rightarrow \left[ \begin{array}{l} 2 \left(\frac{u}{v}\right)^3 S\{\omega(t)\} \\ - \frac{1}{2} \left(\frac{u}{v}\right)^4 S\{\omega'''(t)\} = 2 \left(\frac{u}{v}\right)^5 \end{array} \right] \end{aligned} \quad (28)$$

Applying the property ‘‘Shehu transform of derivative of functions’’ on (28), we get

$$\begin{aligned} & \left[ \begin{array}{l} 2 \left(\frac{u}{v}\right)^3 S\{\omega(t)\} \\ - \frac{1}{2} \left(\frac{u}{v}\right)^4 \left[ \begin{array}{l} \left(\frac{v}{u}\right)^3 S\{\omega(t)\} \\ - \left(\frac{v}{u}\right)^2 \omega(0) \\ - \left(\frac{v}{u}\right) \omega'(0) \\ - \omega''(0) \end{array} \right] = 2 \left(\frac{u}{v}\right)^5 \end{array} \right] \end{aligned} \quad (29)$$

Now using (26) in (29), we have

$$\begin{aligned} & \left[ \begin{array}{l} 2 \left(\frac{u}{v}\right)^3 S\{\omega(t)\} \\ - \frac{1}{2} \left(\frac{u}{v}\right)^4 \left[ \left(\frac{v}{u}\right)^3 S\{\omega(t)\} - 3 \left(\frac{v}{u}\right) \right] = 2 \left(\frac{u}{v}\right)^5 \end{array} \right] \\ & \Rightarrow \left[ \begin{array}{l} \left[ 2 \left(\frac{u}{v}\right)^3 - \frac{1}{2} \left(\frac{u}{v}\right) \right] S\{\omega(t)\} \\ = 2 \left(\frac{u}{v}\right)^5 - \frac{3}{2} \left(\frac{u}{v}\right)^3 \end{array} \right] \\ & \Rightarrow \left[ \begin{array}{l} \left[ \frac{4 \left(\frac{u}{v}\right)^3 - \left(\frac{u}{v}\right)}{2} \right] S\{\omega(t)\} \\ = 2 \left(\frac{u}{v}\right)^5 - \frac{3}{2} \left(\frac{u}{v}\right)^3 = \left[ \frac{4 \left(\frac{u}{v}\right)^5 - 3 \left(\frac{u}{v}\right)^3}{2} \right] \end{array} \right] \\ & \Rightarrow \left[ \begin{array}{l} S\{\omega(t)\} = \left[ \left(\frac{u}{v}\right)^2 - \frac{2u^2}{(4u^2-v^2)} \right] \\ = \left[ \left(\frac{u}{v}\right)^2 + \frac{2u^2}{v^2-4u^2} \right] \end{array} \right] \end{aligned} \quad (30)$$

Taking inverse Shehu transform of both sides of (30), we get the required solution of (25) with (26) as

$$\begin{aligned} & \left[ \begin{array}{l} \omega(t) = S^{-1} \left\{ \left(\frac{u}{v}\right)^2 + \frac{2u^2}{v^2-4u^2} \right\} \\ = S^{-1} \left\{ \left(\frac{u}{v}\right)^2 \right\} + 2S^{-1} \left\{ \frac{u^2}{v^2-4u^2} \right\} \end{array} \right] \\ & \Rightarrow \omega(t) = t + \sinh 2t. \end{aligned}$$

## VI. CONCLUSIONS

In this paper, authors successfully determined the primitive of first kind Volterra integro-differential equation using Shehu transform and four numerical problems solved in numerical problem session of the paper for explaining the complete methodology. The results of numerical problems show that the Shehu transform is very useful integral transform for handling the problem of determining the primitive of first kind Volterra integro-differential equation. In future, Shehu transform can be used for determining the primitive of system of first kind Volterra integro-differential equations.

## REFERENCES

- [1] Aggarwal, S., Chauhan, R., & Sharma, N. (2018). A new application of Mahgoub transform for solving linear Volterra integral equations. *Asian Resonance*, 7(2), 46-48.
- [2] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Application of Mahgoub transform for solving linear Volterra integral equations of first kind. *Global Journal of Engineering Science and Researches*, 5(9), 154-161.
- [3] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). A new application of Aboodh transform for solving linear Volterra integral equations. *Asian Resonance*, 7(3), 156-158.
- [4] Aggarwal, S., Gupta, A. R., & Sharma, S. D. (2019). A new application of Shehu transform for handling Volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 7(4), 439-445.
- [5] Aggarwal, S., Chauhan, R., & Sharma, N. (2018). Application of Elzaki transform for solving linear Volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 6(12), 3687-3692.
- [6] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Application of Aboodh transform for solving linear Volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 6(12), 3745-3753.
- [7] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Solution of linear Volterra integral equations of second kind using Mohand transform. *International Journal of Research in Advent Technology*, 6(11), 3098-3102.
- [8] Aggarwal, S., Chauhan, R., & Sharma, N. (2018). A new application of Kamal transform for solving linear Volterra integral equations. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 7(4), 138-140.
- [9] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Solution of linear Volterra integro-differential equations of second kind using Mahgoub transform. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 7(5), 173-176.
- [10] Aggarwal, S., & Gupta, A. R. (2019). Solution of linear Volterra integro-differential equations of second kind using Kamal transform. *Journal of Emerging Technologies and Innovative Research*, 6(1), 741-747.
- [11] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Application of Aboodh transform for solving linear Volterra integro-differential equations of second kind. *International Journal of Research in Advent Technology*, 6(6), 1186-1190.
- [12] Chauhan, R., & Aggarwal, S. (2018). Solution of linear partial integro-differential equations using Mahgoub transform. *Periodic Research*, 7(1), 28-31.
- [13] Gupta, A. R., Aggarwal, S., & Agrawal, D. (2018). Solution of linear partial integro-differential equations using Kamal transform. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 7(7), 88-91.
- [14] Singh, G. P., & Aggarwal, S. (2019). Sawi transform for population growth and decay problems. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 8(8), 157-162.
- [15] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Solution of population growth and decay problems by using Mohand transform. *International Journal of Research in Advent Technology*, 6(11), 3277-3282.
- [16] Aggarwal, S., Gupta, A. R., Asthana, N., & Singh, D. P. (2018). Application of Kamal transform for solving population growth and decay problems. *Global Journal of Engineering Science and Researches*, 5(9), 254-260.
- [17] Aggarwal, S., Sharma, S. D., & Gupta, A. R. (2019). Application of Shehu transform for handling growth and decay problems. *Global Journal of Engineering Science and Researches*, 6(4), 190-198.
- [18] Aggarwal, S., Singh, D. P., Asthana, N., & Gupta, A. R. (2018). Application of Elzaki transform for solving population growth and decay problems. *Journal of Emerging Technologies and Innovative Research*, 5(9), 281-284.
- [19] Aggarwal, S., Gupta, A. R., Singh, D. P., Asthana, N., & Kumar, N. (2018). Application of Laplace transform for solving population growth and decay problems. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 7(9), 141-145.
- [20] Aggarwal, S., Pandey, M., Asthana, N., Singh, D. P., & Kumar, A. (2018). Application of Mahgoub transform for solving population growth and decay problems. *Journal of Computer and Mathematical Sciences*, 9(10), 1490-1496.
- [21] Aggarwal, S., Sharma, N., & Chauhan, R. (2020). Duality relations of Kamal transform with Laplace, Laplace-Carson, Aboodh, Sumudu, Elzaki, Mohand and Sawi transforms. *SN Applied Sciences*, 2(1), 135.
- [22] Aggarwal, S., & Bhatnagar, K. (2019). Dualities between Laplace transform and some useful integral transforms. *International Journal of Engineering and Advanced Technology*, 9(1), 936-941.
- [23] Chauhan, R., Kumar, N., & Aggarwal, S. (2019). Dualities between Laplace-Carson transform and some useful integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, 8(12), 1654-1659.
- [24] Aggarwal, S., & Gupta, A. R. (2019). Dualities between Mohand transform and some useful integral transforms. *International Journal of Recent Technology and Engineering*, 8(3), 843-847.
- [25] Aggarwal, S., & Gupta, A. R. (2019). Dualities between some useful integral transforms and Sawi transform. *International Journal of Recent Technology and Engineering*, 8(3), 5978-5982.
- [26] Aggarwal, S., Bhatnagar, K., & Dua, A. (2019). Dualities between Elzaki transform and some useful integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, 8(12), 4312-4318.
- [27] Aggarwal, S., Sharma, N., Chaudhary, R., & Gupta, A. R. (2019). A comparative study of Mohand and Kamal transforms. *Global Journal of Engineering Science and Researches*, 6(2), 113-123.
- [28] Aggarwal, S., Mishra, R., & Chaudhary, A. (2019). A comparative study of Mohand and Elzaki transforms. *Global Journal of Engineering Science and Researches*, 6(2), 203-213.
- [29] Aggarwal, S., & Sharma, S. D. (2019). A comparative study of Mohand and Sumudu transforms. *Journal of Emerging Technologies and Innovative Research*, 6(3), 145-153.
- [30] Aggarwal, S., & Chauhan, R. (2019). A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*, 7(1), 520-529.
- [31] Aggarwal, S., & Chaudhary, R. (2019). A comparative study of Mohand and Laplace transforms. *Journal of Emerging Technologies and Innovative Research*, 6(2), 230-240.
- [32] Aggarwal, S., Gupta, A. R., & Kumar, A. (2019). Elzaki transform of error function. *Global Journal of Engineering Science and Researches*, 6(5), 412-422.
- [33] Aggarwal, S., & Singh, G. P. (2019). Aboodh transform of error function. *Universal Review*, 10(6), 137-150.
- [34] Aggarwal, S., & GP, S. (2019). Shehu Transform of Error Function (Probability Integral). *Int J Res Advent Technol*, 7, 54-60.
- [35] Aggarwal, S., & Sharma, S. D. (2019). Sumudu transform of error function. *Journal of Applied Science and Computations*, 6(6), 1222-1231.
- [36] Aggarwal, S., Gupta, A. R., & Kumar, D. (2019). Mohand transform of error function. *International Journal of Research in Advent Technology*, 7(5), 224-231.
- [37] Aggarwal, S., & Singh, G. P. (2019). Kamal transform of error function. *Journal of Applied Science and Computations*, 6(5), 2223-2235.
- [38] Aggarwal, S., Gupta, A. R., Sharma, S. D., Chauhan, R., & Sharma, N. (2019). Mahgoub transform (Laplace-Carson transform) of error function. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 8(4), 92-98.
- [39] Aggarwal, S., Singh, A., Kumar, A., & Kumar, N. (2019). Application of Laplace transform for solving improper integrals whose integrand consisting error function. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(2), 1-7.

- [40] Aggarwal, S., Sharma, N., Chauhan, R., Gupta, A. R., & Khandelwal, A. (2018). A new application of Mahgoub transform for solving linear ordinary differential equations with variable coefficients. *Journal of Computer and Mathematical Sciences*, 9(6), 520-525.
- [41] Aggarwal, S., & Sharma, S. D. (2019). Application of Kamal transform for solving Abel's integral equation. *Global Journal of Engineering Science and Researches*, 6(3), 82-90.
- [42] Aggarwal, S., & Gupta, A. R. (2019). Sumudu transform for the solution of Abel's integral equation. *Journal of Emerging Technologies and Innovative Research*, 6(4), 423-431.
- [43] Aggarwal, S., Sharma, S. D., & Gupta, A. R. (2019). A new application of Mohand transform for handling Abel's integral equation. *Journal of Emerging Technologies and Innovative Research*, 6(3), 600-608.
- [44] Aggarwal, S., & Sharma, S. D. (2019). Solution of Abel's integral equation by Aboodh transform method. *Journal of Emerging Technologies and Innovative Research*, 6(4), 317-325.
- [45] Aggarwal, S., & Gupta, A. R. (2019). Shehu Transform for Solving Abel's Integral Equation. *Journal of Emerging Technologies and Innovative Research*, 6(5), 101-110.
- [46] Aggarwal, S., Chauhan, R., & Sharma, N. (2018). Mohand transform of Bessel's functions. *International Journal of Research in Advent Technology*, 6(11), 3034-3038.
- [47] Aggarwal, S., Gupta, A. R., & Agrawal, D. (2018). Aboodh transform of Bessel's functions. *Journal of Advanced Research in Applied Mathematics and Statistics*, 3(3), 1-5.
- [48] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Mahgoub transform of Bessel's functions. *International Journal of Latest Technology in Engineering, Management & Applied Science*, 7(8), 32-36.
- [49] Aggarwal, S. (2018). Elzaki transform of Bessel's functions. *Global Journal of Engineering Science and Researches*, 5(8), 45-51.
- [50] Chaudhary, R., Sharma, S.D., Kumar, N., & Aggarwal, S. (2019). Connections between Aboodh transform and some useful integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, 9(1), 1465-1470.
- [51] Aggarwal, S., Chauhan, R., & Sharma, N. (2018). Application of Elzaki transform for solving linear Volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 6(12), 3687-3692.
- [52] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Application of Kamal transform for solving linear Volterra integral equations of first kind. *International Journal of Research in Advent Technology*, 6(8), 2081-2088.
- [53] Aggarwal, S., Asthana, N. & Singh, D.P. (2018). Solution of population growth and decay problems by using Aboodh transform method. *International Journal of Research in Advent Technology*, 6(10), 2706-2710.
- [54] Aggarwal, S., & Bhatnagar, K. (2019). Sadik transform for handling population growth and decay problems. *Journal of Applied Science and Computations*, 6(6), 1212-1221.
- [55] Aggarwal, S., & Sharma, S.D. (2019). Sadik transform of error function (probability integral). *Global Journal of Engineering Science and Researches*, 6(6), 125-135.
- [56] Aggarwal, S., Gupta, A.R., & Sharma, S.D. (2019). Application of Sadik transform for handling linear Volterra integro-differential equations of second kind. *Universal Review*, 10(7), 177-187.
- [57] Aggarwal, S., & Bhatnagar, K. (2019). Solution of Abel's integral equation using Sadik transform. *Asian Resonance*, 8(2), (Part-1), 57-63.
- [58] Aggarwal, S. (2019). A comparative study of Mohand and Mahgoub transforms. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(1), 1-7.
- [59] Aggarwal, S. (2018). Kamal transform of Bessel's functions. *International Journal of Research and Innovation in Applied Science*, 3(7), 1-4.
- [60] Chauhan, R., & Aggarwal, S. (2019). Laplace transform for convolution type linear Volterra integral equation of second kind. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(3&4), 1-7.
- [61] Sharma, N., & Aggarwal, S. (2019). Laplace transform for the solution of Abel's integral equation. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(3&4), 8-15.
- [62] Aggarwal, S., & Sharma, N. (2019). Laplace transform for the solution of first kind linear Volterra integral equation. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(3&4), 16-23.
- [63] Mishra, R., Aggarwal, S., Chaudhary, L., & Kumar, A. (2020). Relationship between Sumudu and some efficient integral transforms. *International Journal of Innovative Technology and Exploring Engineering*, 9(3), 153-159.
- [64] Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. *International Journal of Analysis and Applications*, 17(2), 167-190.