On The Diophantine Equation $223^x + 241^y = z^2$

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***Abstract*: In this article, authors discussed the existence of solution of Diophantine equation $223^x + 241^y = z^2$, where x, y, z are non-negative integers. Results show that the consider Diophantine equation of study has no non-negative integer solution.**

***Keywords:* Prime number; Diophantine equation; Solution, Integers.**

***Mathematics Subject Classification:* 11D61, 11D72, 11D45.**

I. INTRODUCTION

Diophantine equations are those equations which are to be solved in integers. Diophantine equations are very important equations of theory of numbers and have many important applications in algebra, analytical geometry and trigonometry [4, 6]. These equations give us an idea to prove the existence of irrational numbers. Acu [1] studied the Diophantine equation $2^x + 5^y = z^2$ and proved that $\{x = 3, y = 0, z = 3\}$ and $\{x = 2, y = 1, z = 3\}$ are the solutions of this equation. Kumar et al. [2] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [3] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution. Rabago [5] discussed the open problem given by B. Sroysang. He showed that the Diophantine equation $8^x + p^y = z^2$, where x, y, z are positive integers has only three solutions namely $\{x = 1, y = 1, z = 5\}$, $\{x = 2, y = 1, z = 9\}$ and $\{x = 3, y = 1, z = 23\}$ for $p = 17$. The Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [7-8]. He proved that these equations have a unique non-negative integer solution namely $\{x = 1, y = 0, z = 3\}$. Sroysang [9] proved that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

The main aim of this article is to discuss the existence of solution of Diophantine equation $223^x + 241^y = z^2$, where x, y, z are non-negative integers.

II. PRELIMINARIES

Lemma: 1 The Diophantine equation $223^x + 1 = z^2$, where x, z are non-negative integers, has no solution in non-negative integers.

Proof: Since 223 is an odd prime so 223^x is an odd number for all non-negative integer x.

$\Rightarrow 223^x + 1 = z^2$ is an even number for all non-negative integer x.

$\Rightarrow z$ is an even number.

$\Rightarrow z^2 \equiv 0(mod 3)$ or $z^2 \equiv 1(mod 3)$ (1)

Now, $223 \equiv 1(mod 3)$

$\Rightarrow 223^x \equiv 1(mod 3)$, for all non-negative integer x

$\Rightarrow 223^x + 1 \equiv 2(mod 3)$, for all non-negative integer x

$\Rightarrow z^2 \equiv 2(mod 3)$ (2)

Equation (2) contradicts equation (1). Hence Diophantine equation $223^x + 1 = z^2$ has no non-negative integer solution.

Lemma: 2 The Diophantine equation $241^y + 1 = z^2$, where y, z are for all non-negative integers, has no solution in non-negative integers.

Proof: Since 241 is an odd prime so 241^y is an odd number for all non-negative integer y.

$\Rightarrow 241^y + 1 = z^2$ is an even number for all non-negative integer y

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod 3)$ or $z^2 \equiv 1(mod 3)$ (3)

Now, $241 \equiv 1(mod 3)$

$\Rightarrow 241^y \equiv 1(mod 3)$, for all non-negative integer y

$\Rightarrow 241^y + 1 \equiv 2(mod 3)$, for all non-negative integer y

$\Rightarrow z^2 \equiv 2(mod 3)$ (4)

Equation (4) contradicts equation (3). Hence Diophantine equation $241^y + 1 = z^2$ has no non-negative integer solution.

Main Theorem: The Diophantine equation $223^x + 241^y = z^2$, where x, y, z are non-negative integers, has no solution in non-negative integers.

Proof: There are three cases:

Case: 1 If $x = 0$ then the Diophantine equation $223^x + 241^y = z^2$ becomes

$1 + 241^y = z^2$, which has no non-negative integer solution by lemma 2.

Case: 2 If $y = 0$ then the Diophantine equation $223^x + 241^y = z^2$ becomes $223^x + 1 = z^2$, which has no non-negative integer solution by lemma 1.

Case: 3 If x, y are positive integers, then $223^x, 241^y$ are odd numbers.

$\Rightarrow 223^x + 241^y = z^2$ is an even number

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod3)\ or\ z^2 \equiv 1(mod3)$ (5)

Now, $223 \equiv 1(mod3)$

$\Rightarrow 223^x \equiv 1(mod3)\ and\ 241 \equiv 1(mod3)$

$\Rightarrow 223^x \equiv 1(mod3)\ and\ 241^y \equiv 1(mod3)$

$\Rightarrow 223^x + 241^y \equiv 2(mod3)$

$\Rightarrow z^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5). Hence Diophantine equation $223^x + 241^y = z^2$ has no non-negative integer solution.

III. CONCLUSION

In this article, authors successfully discussed the solution of Diophantine equation $223^x + 241^y = z^2$, where x, y, z are non-negative integers and determined that this equation has no non-negative integer solution.

CONFLICT OF INTERESTS

Authors state that this paper has no conflict of interest.

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