On Theory of Envelopes and its Applications

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Abstract: **In this work, we formulate the renormalization group (RG) method for global analysis using the classical theory of envelope. Actually, what the RG method does is to construct an approximate but global solution from the ones with a local nature which was obtained in the perturbation theory. Finally, we give some applications of theory of envelopes.**

Keywords: **Differential equations, envelopes, renormalization group.**

I. INTRODUCTION

Most differential equations can not be solved exactly and can only be handled by various perturbation or asymptotic analysis. This is why perturbation theory and asymptotic analysis constitute such an important topic in mathematical physics and have applications to various natural sciences [9]. Perturbation theory usually refers to collection of iterative methods for the systematic analysis of global behaviour of differential equations. It usually proceeds by an identification of a small parameter, say ϵ, in the problem such that when $\epsilon = 0$, the problem is exactly solvable. The global solution to the problem then can be studied via local analysis about ϵ and solution can be expressed by a regular perturbation expansion:

$$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \cdots . \quad (0.1)$$

Such a series is called a perturbation series where $x_n(t)$ can always be computed in terms of $x_0, x_1, \cdots, x_n-1$ as long as the $\epsilon = 0$ problem is exactly solvable. Usually when ϵ is small, it's expected that only a few terms of the perturbation series are enough for a well approximated solution.

When the highest order derivative of a given differential equation is multiplied by a small parameter, ϵ, then the equation lead to narrow regions of rapid variation called boundary layers. Such cases constitute yet another class of problems where regular perturbation theory fails. In cases where the small parameter, $\epsilon \to 0$, boundary-layer techniques can be employed.

The recently developed of renormalization group (RG) method introduced by [1], opened a new direction of research in non-linear dynamics. They showed that RG can be used as that global and asymptotic analysis tool for ODEs and PDEs. What makes the method so powerful is it starts with a regular perturbation expansion and substitutes in the equation, then uses the renormalization transform that will deals with the secular terms and applies RG condition to obtain a valid solution.

II. THEORY OF ENVELOPES

Let $\{C_T\}_T$ be a family of curves with parameter T in the x, y plane, where C_T is represented by the equation

$$f(x, y, T) = 0. \quad (0.2)$$

Now we assume that the family of the curves has an envelops E which is also in the form

$$G(x, y) = 0. \quad (0.3)$$

Our aim is to obtain $G(x, y)$ from $f(x, y, T)$. Suppose that both E and a curve C_{T0} have the same tangent line at $(x, y) = (x_0, y_0)$, i.e., (x_0, y_0) is the point of tangency. This implies that x_0 and y_0 are functions of T_0 which can be express as $x_0 = \Phi(T_0)$, $y_0 = \Psi(T_0)$, and $G(x_0, y_0) = 0$. Conversely, for every point on E, say (x_0, y_0), there exists a parameter T_0. Now we can get T_0 as a function of (x_0, y_0) and $G(x, y)$ can be express as

$$f(x, y, T(x, y)) = G(x, y).$$

Since x_0 and y_0 are both functions of T_0, then $T_0(x_0, y_0)$ can be obtained by defining the tangent line of E at point (x_0, y_0) and that of C_{T0} at the same point.

For E at point (x_0, y_0) is : $\Psi(T_0)(x - x_0) - \Phi(T_0)(y - y_0) = 0$ and

For C_{T0} at point (x_0, y_0) is : $Fx(x_0, y_0, T_0)(x - x_0) + Fy(x_0, y_0, T_0)(y - y_0) = 0$,

Where both Fx and Fy are partial derivatives of F with respect to x and y respectively. But since they are at the same point, then the above equations must produce the same line as well. Therefore, we have

$$Fx(x_0, y_0, T_0)\Phi(T_0) + Fy(x_0, y_0, T_0)\Psi(T_0) = 0.$$

Similarly, if we differentiate the function $F(x(T_0), y(T_0), T_0) = 0$ partially with respect to parameter T_0, we get

$$Fx(x_0, y_0, T_0)\Phi(T_0) + Fy(x_0, y_0, T_0)\Psi(T_0) + FT_0(x_0, y_0, T_0) = 0,$$

then $F_{T0}(x_0, y_0, T_0)$ is identical to $\frac{\partial F(x_0, y_0, T_0)}{\partial_{T0}} = 0$, that is ,

$$F_{T0}(x_0, y_0, T_0) \equiv \frac{\partial F(x_0, y_0, T_0)}{\partial_{T0}} = 0.$$

To get the relation between x_0 and y_0, we eliminate the parameter T_0, and by transforming

$(x_0, y_0) \rightarrow (x, y)$ we get

$$G(x, y) = F(x, y, T_0(x, y)) = 0.$$

If $x = g(y, T)$ is the family of curves, then $\frac{\partial F(x_0, y_0, T_0)}{\partial T_0} = 0$ implies $\frac{\partial F}{\partial T_0} = 0$, where the envelope is $x = g(y, T_0(y))$.

Also, we can get both E and a set of singularities of the curves $\{C_T\}_T$ from the equation $G(x, y) = 0$ since $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ satisfy $\frac{\partial F(x_0, y_0, T_0)}{\partial T_0} = 0$.

Application of the Envelope

In order to see what this envelope is all about, let's look at a function that is bounded local but not globally.

Consider the function.

$$x = g(y, T) = e^{-\epsilon^3 T}\ (\epsilon_2(y - T) - 1) + e^{-\epsilon y}$$

Clearly this function is bounded locally i.e when $\epsilon = 0$. For $y - T \rightarrow \infty$ then $g(y, T)$ is unbounded. We can now get the envelope E of the given curve C_T from the condition $\frac{\partial g}{\partial T} = 0$.

Now $\frac{\partial g}{\partial T} = x_T = -\epsilon^2\ e^{-\epsilon^2 T} - \epsilon^2\ e^{-\epsilon^2 T}\ (\epsilon^2(y - T) - 1) = 0$, this implies that

$$y = T$$

Where the parameter T is on the y-coordinate of the point of tangency of the curves, C_T and envelope E. This implies that $x = g(y, y) = e^{-\epsilon y} - e^{-\epsilon^2 y}$ and this envelope is bounded even for $y \rightarrow \infty$. Hence, $g(y, y) = e^{-\epsilon y} - e^{-\epsilon^2 y}$ is an envelope with global nature derived from the curves, $g(y, T)$ that bounded only locally.

BVP using Theory of Envelopes

Let's consider the equation in [10] which is the second order linear differential equation of the form

$$\epsilon\ \frac{\delta^2 \xi}{\delta x^2} + \frac{\delta \xi}{\delta x} + \epsilon\ \frac{\delta \xi}{\delta x} + \xi = 0 \qquad (0.4)$$

Equipped with boundary conditions $\xi(0) = 0$ and $\xi(1) = 1$.

This can be interpret as a boundary-layer problem. Using the [10], we obtained the exact solution as

$$\xi(x) = \frac{e^{-\chi} - e^{-\frac{\chi}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}} \qquad (0.5)$$

Next is to assign the variable $\chi = \epsilon X$ and $\xi(x) = Y(X)$, which when differentiate we get

$$\frac{1}{\epsilon}\ \frac{dY}{dX} = \frac{d\xi}{dx} \text{ and } \frac{1}{\epsilon^2}\ \frac{d^2 Y}{dX^2} = \frac{d^2 \xi}{dx^2}$$

Substitute this relations in Equation (0.4) , then it becomes

$$\epsilon\left(\frac{1}{\epsilon^2}\ \frac{d^2 Y}{dX^2}\right) + (1 + \epsilon)\frac{1}{\epsilon}\ \frac{dY}{dX} + Y = 0$$

which gives

$$\frac{d^2 Y}{dX^2} + \frac{dY}{dX} = -\epsilon\left(\frac{dY}{dX}\right) + Y \qquad (0.6)$$

Then we apply the naive perturbation expansion

$$Y(X) = Y_0(X) + \epsilon Y_1(X) + \epsilon^2 Y_2(X) + \cdots \qquad (0.7)$$

to the Equation (0.6) , to obtain

$$\ddot{Y}_0 + \ddot{Y}_1 + \cdots \dot{Y}_0 + \dot{Y}_1 + \cdots = -(\dot{Y}_0 + \dot{Y}_1 + \cdots + Y_0 + Y_1 + \cdots). \qquad (0.8)$$

After equating the corresponding orders of ϵ, we get

$O(1) : \ddot{Y}_0 + \dot{Y}_0 = 0$ (0.9)

$O(\epsilon) : \ddot{Y}_1 + \dot{Y}_1 = -(\dot{Y}_0 + Y_0)$, and so on (0.10)

with the boundary condition $Y(X) = Y_0(X_0) = A_0$ (0.11)

for any arbitrary constant X_0 and A_0 as function of X_0. The solution of (0.9) is

$$Y_0(X) = A + Be^{-X}$$

and by applying the boundary conditions we have

$$Y_0(X) = A_0 - B0e^{-(X - X_0)}. \qquad (0.12)$$

For the second Equation of (0.9), we have the solution

$$Y_1(X) = -A_0(X - X_0) - (B_0 + C_0)\ e^{-(X - X_0)} - 1). \qquad (0.13)$$

By substituting Equations (0.12) and (0.13) in (0.7), we obtain the curves describe as

$$Y(X, X_0) = A_0 - B_0 e^{-(X - X_0)} - \epsilon\left(A_0(X - X_0) + (B_0 + C_0)e^{-(X - X_0)} - 1\right)$$

Now we are to define the renormalization constants A and B such that A0 and B0 would be absorb from the above curves. This can be seen as

$A = A_0 + \epsilon(B_0 + C_0)$

$B = B_0 + \epsilon(B_0 + C_0)$

this implies that $Y(X, X_0) = A - Be^{-(X - X_0)} - \epsilon A(X - X_0) + O(\epsilon^2)$. (0.14)

By changing Equation (0.14) in to the given coordinate, we obtain

$$\xi(x, x_0) = A - Be^{\frac{-(X - X_0)}{x}} - A(x, x_0) + O(\epsilon^2),$$

because of the relation $x_0 = \frac{x_0}{\epsilon}$. Note that, the family of functions is $\{Y(X, X_0)\}x_0$. Now we can derive the envelope YE(X) from $\{Y(X, X_0)\}x_0$ and both has the common tangent line at $X = X_0$. To do that, we use the condition derived in the previous section,

$$\frac{\partial Y}{\partial X_0}\Big|_{X = X_0} = 0 \qquad (0.15)$$

where Y (X, X) will be the envelopes, $Y_E(X)$. Using condition (0.15) in the Equation (0.14), we get

$$\frac{\partial Y}{\partial X_0} = \frac{dA}{dX_0} - \mathrm{B}e^{-(X-X_0)} - \frac{dB}{dX_0} e^{-(X-X_0)} + \epsilon \mathrm{A} - \epsilon\frac{dA}{dX_0}(\mathrm{X} - \mathrm{X}_0)\Big|_{X=X_0} = 0$$

$$\frac{dA}{dX} + \epsilon\mathrm{A} = 0 \text{ and } \frac{dB}{dX} + \mathrm{B} = 0 \qquad (0.16)$$

using separation of variables, yields

$A(X) = \alpha e^{-\epsilon X}$ and $B(X) = \beta e^{-\epsilon X}$

where $\alpha = e^{\epsilon 1}$ and $\beta = e^{\epsilon 2}$ are constants. This implies that, Equation (0.14) becomes the required envelope in form

$$Y_E(X) = Y(X, X) = A(X) - B(X) = \alpha e^{-(\epsilon X)} - \beta e^{-X} \qquad (0.17)$$

And in terms of the given unknown the envelope becomes

$$\xi_E(x) = \alpha e^{-x} - \beta e^{-\frac{x}{\epsilon}} \qquad (0.18)$$

with boundary conditions $\xi(0) = 0$ and $\xi(1) = 1$. To obtain α and β,

$$\xi(0) = \alpha e^{-(0)} - \beta e^{-\frac{0}{\epsilon}} = 0 \quad \alpha = \beta \text{ and} \qquad (0.19)$$

$\xi(1) = \alpha e^{-(1)} - \beta e^{-\frac{1}{\epsilon}} = 1$ and by using (0.19) we get $\alpha e^{-(1)} - \alpha e^{-\frac{1}{\epsilon}} = 1$, i.e.,

$$\alpha = \frac{1}{e^{-1} - e^{-\frac{1}{\epsilon}}} \qquad (0.20) \text{ this implies}$$

$\alpha = \beta = \frac{1}{e^{-1} - e^{-\frac{1}{\epsilon}}}$. Therefore, $\xi_E(x) = \frac{e^{-x} - e^{-\frac{x}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}}$

Hence, the envelope $\xi_E(x)$ coincides with the exact solution $\xi(x)$ i.e., equation (0.5), and satisfy both the inner and outer boundary conditions together.

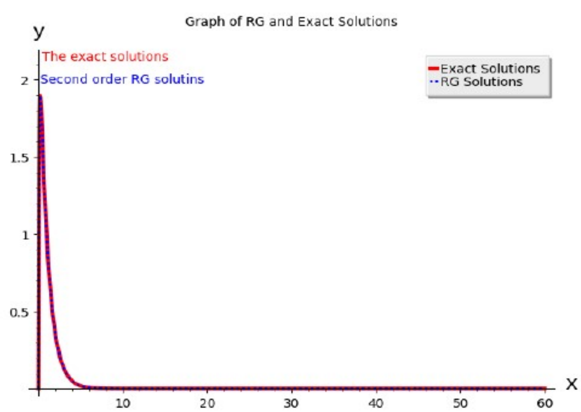

Figure 1: This is the graph solutions of exact (0.5) and RG method (0.18) using theory of envelope to second order linear differential equation (0.4) for small parameter, $\epsilon = \frac{1}{10}$.

Concluding Remarks In this work, we discussed how the Renormalization Group methods works in approximating the solution of differential equation using theory of envelopes and we show logically how the secular terms that arise in the naive perturbation expansion can be eliminate using "renormalization transform ". By this method of RG, we obtained detailed analytical results for singularly perturbed problems and compare then with the exact solution of the same problem.

Renormalization Group Method is clear in theory but difficult in practice. It is no doubt that the proposed method can be applied to many linear and nonlinear differential equations. For the future work, we want to propose how to apply this method in wireless market intelligence as it's described in [11]. Finally here is a question left to a reader. Can you reformulate the undamped nonlinear oscillator described by Duffing's equation .

$$\frac{d^2x(t)}{dt^2} + x(t) + \epsilon\, x(t)^3 = 0$$

Using theory of envelopes up to including order O () and what can you deduce?.

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