Conditions Implying Convexoidity and Normaloidity

G.M. Kariuki, M. Kavila
Department of Mathematics and Actuarial Science Kenyatta University P.O.Box 43844-00100, Nairobi, Kenya

Abstract: Let $B(H)$ denote the algebra of bounded linear operators on a Hilbert space $H$ into itself. Our task in this note is to prove conditions that imply convexoidity and normaloidity. It is shown among other results that if $T$ is normaloid then $T^k$ is normaloid for $k \in \mathbb{N}$.

AMS Subject classification: 47B47, 47A30, 47B20

Key words: normal, hyponormal, convexoid and normaloid operators.

I. INTRODUCTION

Let $B(H)$ denote the algebra of bounded operators on a complex Hilbert space $H$ into itself. Istrăţescu [3] proved the following result.

Theorem A. Let $T \in B(H)$ be a hyponormal operator. Then $T - \mu$ is also hyponormal for any $\mu \in \mathbb{C}$.

Sheth [4] also proved the following result involving self-adjoint operators.

Theorem B. Let $T \in B(H)$. Then $T$ is normaloid if it is self-adjoint.

In this paper we will outline an alternative proof to Theorem B and also give other conditions that imply normaloidity and convexoidity.

The following lemma by Blumenson [1] will also act as a stepping stone to the main theorem in this paper involving normaloid operators.

Lemma C. Let $T \in B(H)$. Then

(i) $\frac{1}{2} ||T|| \leq w(T) \leq ||T||$
(ii) $r(T) \leq w(T) \leq ||T||$

II. NOTATION AND TERMINOLOGY

Given an operator $T \in B(H)$, we shall denote the spectral radius and the numerical radius of $T$ by $r(T)$ and $w(T)$ respectively.

$r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$

$w(T) = \sup \{ ||x|| : \lambda \in W(T) \}$ where W(T) and $\sigma(T)$ denote the numerical range of T and the spectrum of T respectively given by:

$W(T) = \{ \langle Tx, x \rangle : ||x|| = 1, x \in H \}$

$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}$

An operator $T$ is said to be:

- Self-adjoint if $T = T^*$
- Normal if $T^*T = TT^*$
- Hyponormal if $T^* \geq TT^*$
- Normaloid if $\|T\| = r(T), \|T^n\| = \|T\|^n$, and $\|T\| = w(T)$
- Convexoid if $\|(T - \mu)^{-1}\| \leq \frac{1}{d(\mu, \sigma(T))}$ for all $\mu \notin \sigma(T)$
- Or $T - \mu$ is spectraloid.

Note that we have the following inclusions of classes of operators:

$\text{hyponormal} \subseteq \text{normaloid}, \text{normaloid} \subseteq \text{spectraloid}$ and $\text{normal} \subseteq \text{hyponormal}$.

III. RESULTS

Theorem 1. If $T$ is normaloid operator, then $T^k$ is also normaloid for $k \in \mathbb{N}$.

Proof. Since $T$ is normaloid, it implies $r(T) = ||T||$.

Assume $r(T^k) = ||T^k||$ holds.

Now, $r(T^{k+1}) = \sup \{ ||x|| : \lambda \in \sigma(T^{k+1}) \}$

By Lemma C, $r(T^{k+1}) = r(T^{k}) \leq ||T^{k+1}||$

$r(T^{k+1}) = r(T)^k \leq \lim_{m \to \infty} ||T^m^{(k+1)}||^{\frac{1}{m}}$

Thus, $r(T^{k+1}) \leq ||T^{k+1}||$

Now, $||T^{k+1}|| \leq r(T^{k+1})$ always holds.

$r(T^{k+1}) = ||T^{k+1}||$

Hence $T^k$ is normaloid by induction.

Remark 1. We now show conditions implying convexoidity and normaloidity by establishing corollaries to Theorem A and give an alternative prove to Theorem B.

Corollary I. Every hyponormal operator is convexoid.

Proof. Let $T$ be hyponormal. Then $T^*T \geq TT^*$ always holds. It follows from Theorem A that $T - \mu$ is hyponormal.

Since every hyponormal operator is normaloid, we have $T - \mu$ is normaloid.

Since every normaloid operator is spectraloid, it implies $T - \mu$ is spectraloid.
Thus, $T$ is convexoid.

**Corollary 2.** Every normal operator is convexoid.

**Proof.** Let $T$ be normal. Then $T^*T = TT^*$.

Since every normal operator is hyponormal, it follows that $T$ is hyponormal.

By Corollary 1, $T$ is convexoid.

**IV. ALTERNATIVE PROOF TO THEOREM B.**

Let $T$ be self-adjoint. Then $T = T^*$

$\Rightarrow \|T\| = \|T^*\|$. Now,

$\|T\| = \text{Sup} \{ |\langle x, Tx \rangle | : \|x\| = 1 \} = w(T)$.

Thus, $\|T\| = w(T)$. Hence $T$ is normaloid.

**Remark 2.** We conclude by outlining the proof of normality implying normaloidity.

Note that it is known that

$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$ holds for any operator $T$.

**Corollary 3.** Let $T \in B(H)$. Then $T$ is normaloid if it is normal.

**Proof.** Since $T$ is normal, $T^*T = TT^*$. This implies $T^*T$ is self-adjoint.

$\therefore (T^*T)^n = (TT^*)^n = T^{*n}T^n$

Thus $\|T\| = \|T^*T\|^\frac{1}{n} = \lim_{n \to \infty} \|(T^*T)^n\|^\frac{1}{2n}$

$= \lim_{n \to \infty} \|T^{*n}T^n\|^\frac{1}{2n}$

$= \lim_{n \to \infty} \|T^n\|^\frac{1}{n}$

$= r(T)$.

Hence $T$ is normaloid.

**REFERENCES**


