Generalization of Pure-Supplemented Modules

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Abstract: Let R be a ring and M be an R-module. We generalized the concepts pure-lifting and pure-supplemented module and introduce weak distribution with fully invariant. We prove every pure g-lifting is pure g-supplemented module. Let M be a weak distribution pure g-supplemented module, then M/A is pure g-supplemented module for every submodule A of M. Let M = M₁@M₂ be a weakly distributive R-module. Then each M₁, i∈{1, 2} is closed weak g-supplemented if and only if M is closed weak g-supplemented.

Key Words: g-small, g-supplemented, pure-lifting, pure-supplemented, pure g-supplemented, closed weak g-supplemented, Distributive, weak Distributive modules.

I. INTRODUCTION

Throughout this paper R is an associative ring with unity and all modules are unitary R-modules. [12] Sahira M. Yasen and W. Khalid Hasan introduce the concepts pure-module and pure-supplemented module with some conditions. Let M be an R-module, a submodule L of module M is denoted by L ≤ M. Submodule L of M is called essential (large) in M, abbreviated K ≤ M, if for every submodule N of M, L ∩ N implies N = 0. A submodule N of a module M is called small in M, denoted by N≪M, if for every sub module L of M, the equality N + L = M implies L = M. [2] A submodule K of M is called generalized small (g-small) submodule of M denoted by N K M, if for every small submodules T of M with the property M = K + T implies that T = M. Supplemented modules and two other generalizations amply supplemented and weakly supplemented modules were studied by Helmut Zoschinger and he posed their whole structure over discrete valuation rings. After Zoschinger, some variations of supplemented modules were studied. Let M be an R-module and U, V are submodules of M. If M = U + V and V is minimal with respect to property, or equivalently, M = U + V and U ∩ V ≪ V, then V is called a supplement of U in M. M = M is called supplemented if every submodule of M has supplement in M. If M = U + V and U ⊂ V ≪ M, then V is called a weak supplement of U in M. [10] M is called weak supplemented if every submodule of M has weak supplement in M. [12] Let M be an R-module. P is called a g-pure sub module of M if KM ∩ P = KP for every ideal in R. An R-module M is called lifting if for every submodule N of M there is a decomposition M = M₁@M₂ such that M₁ ≤ N and N ∩ M₂ ≪ M. An R-module M is called pure lifting module if for every submodule A of M there exists a pure submodule P of M, P ≤ A such that M = P + X with A ⊂ X ≪ gM. Every g-lifting module is pure g-lifting module. An R-module M is called pure-supplemented module if for given any submodule A of M there exists a pure submodule P of M such that M = A + X iff M = P + X. [2] B. Kosen, C. Nebyen and a. Pakin, introduce the concept g-supplemented module. Let M be an R-module and U, V are submodules of M. If M = U + V and M = U + T with T is essential in V implies T = V, or equivalently, M = U + V and U ∩ V ≪ gM, then V is called g-supplement of U in M.

In this paper we generalized the concepts pure-lifting and pure-supplemented module. The concepts small, e-small, c-small and g-small play a key role in the study of supplemented, weak supplemented pure-supplemented and pure-lifting modules.

Proposition: 1) Every hollow module is pure g-supplemented module.
2) Every lifting module is pure g-supplemented module.
3) Every pure g-supplemented module is weakly g-supplemented module.

Proof: [12].

Theorem: The following are equivalent for an R-module:

1) M is pure-g-lifting module.
2) Every essential submodule N of M can be written as N = A + K, where A is g-pure in M and K ≪ gM.
3) For every essential submodule N of M there exists a pure g-submodule A of N such that M = A + K and N/A ≪ g M/A.

Proof: 1⇒ 2. Let M be pure-g-lifting module i.e. for every submodule N of M there exists a g- pure submodule P of M, P ≤ N such that M = P + X with N ⊂ X ≪ X. Hence N ⊂ X ≪ gM. We have N = N ⊂ M = N ⊂ (P + X) = N ⊂ T + N ⊂ X = P + (N ⊂ X). If A = P, then K = N ⊂ X with (K ⊂ gM).

2⇒ 3. Let N be an essential submodule of M, since N = A + K, where A is g-pure in M and K ≪ gM. We have M = N + L, therefore M/A = A/K + L/A, this implies A + K + L = M.

Since K ≪ gM, therefore A + K = M and N/A ≪ g M/A.
3 \implies 1. Let N be an essential submodule of M, there exists a pure g-submodule A of N such that M = A + K and \( \frac{N}{A} \ll \frac{M}{A} \) to prove that N \( \cap \) K \ll K. Suppose that N \( \cap \) K + T = K, where T \leq K. Then M = A + K = A + N \( \cap \) K + T implies
\[
\frac{M}{A} = \frac{A + (N + K) + T}{A} = \frac{(N + K) + A + T}{A} = \frac{N + A + T}{A} = \frac{N}{A} + \frac{A + T}{A} = \frac{A + T}{A}.
\]
, since \( \frac{N}{A} \ll \frac{M}{A} \) therefore \( \frac{M}{A} = \frac{A + T}{A} \). Hence A + T = M and T = K. Thus N \( \cap \) K \ll K. //

**Proposition:** Every pure g-lifting module is pure g-supplemented module.

**Proof:** Let M be pure g-lifting module and A be a submodule of M. i.e. there exists a pure g-submodule P of M such that P \( \subseteq \) A such that M = P + X with A \( \cap \) X \ll gM. Suppose that M = A + X then M = P + T, where P \( \subseteq \) A and P \ll gM and
\[
A \cap T \ll gM.
\]
We have A = A \cap M = A \cap (P + T) = A \cap P + A \cap T = P + (A \cap T), then
\[
M = A + X = P + (A \cap T) + X.
\]
Since (A \cap T) \ll gM. (A \cap T),then M = P + T thus
\[
M = P + X, since P \subseteq A such that M = P + X with A \cap X \ll gM.
\]
//

**Proposition:** Let M be an R-module is pure g-supplemented module if and only if for every submodule A of M there exist a pure submodule P of M such that
\[
\frac{A + P}{A} \ll \frac{M}{A} \quad and
\]
\[
\frac{A + P}{A} \ll \frac{M}{A}.
\]

**Proof:** \( \Rightarrow \) Let M be an R-module is pure g-supplemented module i.e. every submodule A of M there exists a pure submodule P of M, P \( \subseteq \) A such that M = A + X iff M = P + X with
\[
A \cap X \ll gX. Let K \leq M and suppose that \frac{A + P}{P} + \frac{K}{P} = \frac{M}{P} then \frac{A + P + K}{P} = \frac{M}{P} implies \frac{A + K}{P} = \frac{M}{P}.
\]
Then A + K = M. Since M is pure g-supplemented, therefore A + K = M = P + X, P \( \subseteq \) A, then
\[
A + X \leq K, implies M \leq K. This shows K = M i.e. \frac{K}{P} = \frac{M}{P}
\]
therefore \( \frac{A + P}{P} \cap \frac{K}{P} \ll \frac{K}{P} \) then
\[
\frac{A + P}{P} \ll \frac{M}{P} \quad if \quad M = A + X \quad (X \subseteq M)
\]
\[
\frac{A + P}{P} \ll \frac{A + X + P}{P} = \frac{M}{P}.
\]
\[
\frac{A + P}{P} \ll \frac{M}{P}.
\]
\[
\frac{A + P}{P} \ll \frac{M}{P}.
\]
prove that N \( \subseteq \) A and N \( \subseteq \) X then M = N + X for X is a submodule of M and that f (f (N)) = N in case f is an epimorphism. Moreover, for any submodules L \( \subseteq \) N of M, we have \( f^{-1}(L) \subseteq f^{-1}(N) \). A module M is called duo module if every essential submodule is fully invariant. \[9\] Let M be an R- module and U \subseteq M. A submodule U is said to be a distributive submodule of M if U = U \( \cap \) X + U \( \cup \) Y for all X, Y \( \subseteq \) M. A module M is called distributive if and only if for every submodules K, L, N of M such that N = (K \( \cap \) L) = (N + K) \( \cap \) (N + L) or N \( \cap \) (K + L) = (N \( \cap \) K) + (N \( \cap \) L). Weakly
distribution module are proper generalization of distributive modules. A submodule \( U \) is said to be a weak distributive submodule of \( M \) if \( U = U \cap X + U + Y \) for all \( X, Y \in M \) such that \( X + Y = M \). A module \( M \) is said to be weakly distributive if for every submodule of \( M \) is a weak distributive submodule of \( M \). A ring \( R \) is weakly distributive if \( R \) is a weakly distributive left \( R \)-module.

**Proposition:** Let \( M \) be a weak distributive pure \( g \)-supplemented module, then \( M/A \) is pure \( g \)-supplemented module for every submodule \( A \) of \( M \).

**Proof:** Let \( X \) be direct summand of \( M \), then \( M = X \oplus Y \) for some \( Y \) submodule of \( M \).

Since \( M = X + Y \), therefore \( \frac{M}{A} = \frac{X}{A} + \frac{Y}{A} \) and \( \frac{U}{A} \leq \frac{M}{A} \).

Since \( M \) is a weak distributive pure \( g \)-supplemented module, \( U = (U \cap X) + (U + Y) \) i.e.

\[
\frac{U}{A} = \left( \frac{U \cap X}{A} \right) + \left( \frac{U + Y}{A} \right) = \left( \frac{U \cap X}{A} \right) + \left( \frac{U}{A} \right) = \frac{U \cap X}{A} + \frac{U}{A} = \frac{U}{A}
\]

with \( \frac{X}{A} \cap \frac{Y}{A} = \{0\} \).

\[
\Rightarrow \frac{M}{A} = \frac{X \cap A}{A} \oplus \frac{Y \cap A}{A}. \quad \text{Hence} \quad \frac{X \cap A}{A} \text{ is a direct summand of} \quad \frac{M}{A}
\]

\[
\Rightarrow \frac{M}{A} = \frac{X \cap A}{A} \oplus \frac{Y \cap A}{A}. \quad \text{Hence} \quad \frac{M}{A} \text{ is a pure \( g \)-supplemented module.} \quad \text{//}
\]

**Proposition:** Let \( A \) be a sub module of \( M \) and \( eA \leq A \) for all \( e^2 = e \in \text{End}_R(M) \) then \( \frac{M}{Y} \) is pure \( g \)-supplemented module. In particular for every fully invariant submodule \( Y \) of \( M \), \( \frac{M}{Y} \) is pure \( g \)-supplemented module.

**Proof:** Let \( X \) be the direct summand of \( M \). Now the projection \( e : M \rightarrow X \), then \( e^2 = e \in \text{End}_R(M) \) and \( eA \leq A \), where \( A \) is submodule of \( M \). Hence \( eA = A \cap X \). Then \( M = X + Y \), for some \( Y \in M \),

\[
A = (A \cap X) + (A \cap Y). \quad \text{Now} \quad \frac{X + A}{A} = \frac{X \oplus (A \cap Y)}{A} \quad \text{and} \quad \frac{Y + A}{A} = \frac{Y \oplus (A \cap X)}{A}.
\]

\[
M = X \oplus Y = (X + A) \oplus (Y + A) = \{X \oplus (A \cap Y)\} + (Y + A).
\]

Then

1. \( \frac{M}{A} = \frac{X \oplus (Y + A)}{A} + \frac{Y + A}{A} \)

2. \( \{X \oplus (A \cap Y)\} \cap (Y + A) = ((X \oplus A) \cap (X \oplus Y)) \cap (Y + A) = \{(X \oplus A) \cap Y\} \cap (X \oplus Y) + \{(X \oplus A) \cap A\} \cap (X \oplus Y) \cap A) = \frac{(A \cap Y) \cap (A \cap A)}{A} = \frac{A \cap Y}{A} \cap (A \cap A) = A \)

\[
\frac{M}{A} = \frac{X \oplus (Y + A)}{A} \oplus \frac{Y + A}{A}. \quad \text{Therefore} \quad \frac{Y + A}{A} \leq \frac{M}{A}.
\]

\[
\text{Hence} \quad \frac{M}{A} \text{ is pure \( g \)-supplemented.} \quad \text{//}
\]

**Theorem:** Let \( M = M_1 \oplus M_2 \) be a weakly distributive \( R \)-module. Then each \( M_i, i \in \{1, 2\} \) is closed weak \( g \)-supplemented \( A \) if and only if \( M \) is closed weak \( g \)-supplemented.

**Proof:** Let \( A \leq M \). Since \( M_i \), \( i \in \{1, 2\} \) is closed weak \( g \)-supplemented \( R \)-modules. Let \( M = M_1 \oplus M_2, M_1, M_2 \) are submodules of \( M \). We have \( A \cap M_i \leq M_i \). Let \( A \cap M_i \leq M_i \), since \( M \) is a weakly distributive \( R \)-module.

We have \( A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2) \leq B \oplus (A \cap M_2) \) in \( M \). Since \( A \leq M \) \( \Rightarrow A = (A \cap M_1) \oplus (A \cap M_2) = B \oplus (A \cap M_2) \), therefore \( A \cap M_2 = B \), thus \( A \cap M_1 \leq M_1\). Similarly \( A \cap M_2 \leq M_2 \). Since \( M_1, M_2 \) are closed weak \( g \)-supplemented \( R \)-modules. Then there are sub modules \( N_1, N_2 \) such that \( M_1 = N_1 + (A \cap M_1) \) and \( N_1 \cap (A \cap M_1) = N_1 \cap A \leq N_1 \). Similarly \( M_2 = N_2 + (A \cap M_2) \) and \( N_2 \cap (A \cap M_2) = N_2 \cap A \leq N_2 \). Put \( N = N_1 \oplus N_2 \). So we get

\[
M = M_1 \oplus M_2 = \{N_1 + (A \cap M_1)\} \oplus \{N_2 + (A \cap M_2)\}
\]

\[
= (N_1 \oplus N_2) + ((A \cap M_1) \oplus (A \cap M_2))
\]

\[
= (N_1 \oplus N_2) + (A \cap (M_1 \oplus M_2))
\]

\[
= (N_1 \oplus N_2) + (A \cap M) = (N_1 \oplus N_2) + A
\]

\[
\therefore M = X + A. \quad \text{Since} \quad M \text{ is weakly distributive module. Now} \quad X \cap A = (N_1 \oplus N_2) \cap A
\]

\[
= (N_1 \cap A) \oplus (N_2 \cap A) \leq (M_1 \oplus M_2) = M. \quad \text{Then} \quad X \text{ is weak} \ g \text{-supplement of} \ A \text{ in} \ M.
\]

hence \( M \) is closed weak \( g \)-supplemented. \//

**REFERENCES**


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