Properties of the Rotational Symmetry Group of a Tetrahedron Acting on Its Vertices

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Abstract: **- Platonic solids are 3-dimensional solids that have been a subject of interest to mathematician for years due to their natural unique and regular properties. Euclid (300BC) showed that there are only 5 such solids; tetrahedron, cube, octahedron, dodecahedron and icosahedron. Robert moon in the 20th century expanded the link between the platonic solids and the physical world to the electron shell model in chemistry. To delve into this area, this study deals with the action of rotational symmetry groups of the platonic solids on their respective vertices. The action is showed to be transitive, primitive and semi-regular. The sub-degrees of the action of rotational symmetry group of a tetrahedron on its vertices are $1^{(1)}$and$1^{(3)}$. The rank is 2.**

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I. INTRODUCTION

The Platonic solids in a 3-dimensional space are the convex regular polyhedral which are constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. There are only 5 solids that meet this criteria; tetrahedron, cube, icosahedron, dodecahedron, and icosahedron. In this paper, our interest is reserved to the tetrahedron. A regular tetrahedron is one in which all four faces are equilateral triangles. It has 4 faces, 4 vertices and 6 edges. Due to its regular symmetry we consider a group G of its rotational symmetries acting on its vertices and investigate some of the properties of the action. For more information see ***[4]***.

II. PRELIMINARY NOTES

2.1 Regular Solids

Definition 1: *[4]* A Platonic solid is a convex regular polyhedron with congruent faces, edges and vertices.

Definition 2: A dual of a platonic solid is a similar solid formed by forming vertices above the centres of each face and then connecting them with the adjacent vertices.

2.2 Permutation Groups

Definition 3: Permutations of a set **X** can be defined as a bijection from the set **X** onto itself. All permutations of a set

$$gx = x$$

Theorem 11: *[2]* Let $x \in X$, $|X| > 1$. A transitive group on X is primitive if and only if $\mathbf{G_x}$is a maximal group of **G**.

Theorem 12:(Cauchy-Frobenius theorem) Let **G** act on a non-empty set. Then the number of orbits of **X** is equal to

with n elements form a symmetric group denoted S_n, where the group operation is composition of functions.

Definition 4: Permutation of a finite set is either even or odd depending on whether it can be expressed as the product of an even or odd number of transpositions. Transpositions are of length 2.

Definition 5: A subgroup of $\mathbf{S_n}$, consisting of all even permutations of S_n, is called the Alternating Group of degree n and it is denoted by A_n, and it is of order of $|A_n| = \frac{n!}{2}$

Definition 6: If **G** is a group and **X** is a set, then a (left) group action $\psi$of **G** on X is a function

$$\psi : G \times X \to X: (g, x) \mapsto \psi(g, x)$$

That satisfies the following two axioms(*where we denote* $\psi(g, x)$ *as* gx):

- $ex = x$ for all $x \in X$. (Here, e denotes the identity element of the group **G**)
- $(gh)x = g(hx)$for all $g, h \in G$and all $x \in X$.

Definition 7: Let **G** act on the set **X** and let $x \in X$. Then an orbit of an element X in G is the set

$$Orb_G(x) = \{gx | g \in G\}$$

Definition 8: Let G act on a set X and let $x \in X$. The stabilizer of x is the set given by

$$Stab_G(x) = \{g \in G \mid gx = x\}$$

This is a subgroup of **G** denoted by$\mathbf{G_x}$. $\mathbf{G_x}$is also referred to as the isotropy group.

Theorem 9: (Orbit-Stabilizer theorem) Let **G** act on a set **X** and let $x \in X$ then

$$|Orb_G(x)| = |G : Stab_G(x)|$$

Definition 10: If an action of a group has only 1 orbit, then such an action is said to be transitive. In other words, G is said to act transitively on X if for every pair x, y in X there exist a g $\in$ G such that

$$\frac{1}{|G|}\sum_{g \in G} |fix\,(g)|$$

Where **fix (g)** is the number of elements of X fixed by g $\in$ G is given by;

$$|\mathbf{fix(g)}| = \{\mathbf{g\epsilon G} \mid \mathbf{gx = x}\}$$

Definition 13: *[1]* Let **G** be a group and $\boldsymbol{g\epsilon G}$. Let X be a set with x ϵ X. Then the action of G on X is said to be faithful if gx = x; for all xϵX implies g= xthe identity element in **G**.

Definition14: *[5]* Let a group **G** act transitively on a non-empty set X. The orbits of the isotropy $\boldsymbol{G_x}$ of a point $\boldsymbol{x\epsilon X}$is called suborbits of **G** on **X**. The number,$\boldsymbol{R(G)}$of these sub-orbits is known as the rank of **G** on **X** and the lengths of the suborbits are called the sub-degrees of **G** on **X**.

Definition 15: Let an action of a group **G** on a set **X** be transitive. Then the action is said to be regular if it is also semi-regular. I.e. if

StabG(x) = the identity in G.

Definition 16: *[6]* Suppose that the action of a group G on a non-empty finite set X is transitive. Then a subset $\sigma$of X is referred to as a block of the action if, for each g ϵ G, either g σ = σ or g$\sigma \cap \sigma$=ϕIn particular,ϕ, X and all 1 -element subsets of X are obvious blocks and they are referred to as trivial blocks. If the action of G on the set X has only trivial blocks then action is primitive and imprimitive otherwise.

III. MAIN RESULTS

3.1 Group of Rotational Symmetries of a Tetrahedron

A tetrahedron has 4 vertices, 6 edges and 4 faces. Suppose we let the solid be situated with its centre at the origin in R^3. The group of rotational symmetries in R^3 leaving the tetrahedron invariant will be denoted by G. Its elements are

- The identity, $\boldsymbol{I}$,
- Two rotations through 120^o and 240^oabout each of the four axes joining vertices with centres of opposite faces,
- One rotation through angle 180^o about each of 3joining the midpoints of opposites edges. We note this group isomorphic to $\boldsymbol{A_4}$

These are as illustrated in the figure below

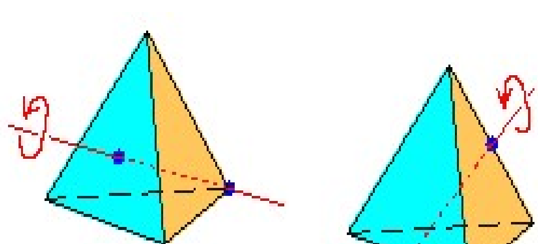

Figure 1: Rotational Symmetries of a Tetrahedron

Remark 3.1: The group of rotational symmetries of a tetrahedron described above is isomorphic to the Alternating group A_4. Therefore;

$$\boldsymbol{G = \{g_1 = I, g_2 = (123)\ , g_3 = (124), g_4 = (132), g_5 = (134), g_6 = (143), g_7 = (124), g_8 = (234), g_9 = (243), g_{10} = (12)(34), g_{11} = (13)(24), g_{12} = (14)(23)\}}$$

Theorem 17: Let $V = \{v_i, v_2, v_3, v_4\}$ be the set of vertices of a tetrahedron where v_i, represents each vertex of the solid. Let G be the group of rotational symmetries of the solid as described above. Then we define the action of G on V by

$$\boldsymbol{g_i(v_j) = v_k}$$

Where $\boldsymbol{g_i\epsilon}$ **G**, $\boldsymbol{i = 1, 2, \dots, 12}$ and $\boldsymbol{v_j, v_k\ \epsilon\ V}$with $\boldsymbol{j, k = 1, 2, 3, 4}$.

Proof:

1. From ***Remark 3.1*** above, $\boldsymbol{g_1}$ is the identity in G, and hence

$$\boldsymbol{g_i(v_i) = (v_i), \forall\ v_i\epsilon\ V}$$

2. Let $\boldsymbol{g_i, g_j\ \epsilon\ G}$, then$\boldsymbol{(g_ig_j)v_k = g_i(g_jv_k)}$

Example 1:

1. Taking $g_1 = I$ and $v_4$then$g_1(v_4) = v_4$
2. Taking $g_2 = (123),\ g_3 = (124)\ \epsilon\ G$ we have,

$$(g_2g_3)v_4 = \{(123)124)v_4$$
$$= (13)(24)v_4$$
$$= v_2$$

On the other hand,

$$g_2(g_3v_4) = (123)\{(124)v_4\}$$
$$= (123)v_1$$
$$= v_2$$

Conclusion 1: Hence the action is well-defined**.**

Lemma 18: *[4]* All dual polyhedrons form the similar symmetrical groups.

Proof: Let P be a platonic solid. By placing points on the centre of every face and then connecting the points to points in neighbouring faces of the original polyhedron will form the dual of the original platonic solid.

Lemma 19: *[3]* The dual of a Tetrahedron is itself.

Proof: From ***Definition 2***, ***Definition 3*** and ***lemma 18*** we observe that the formation of the dual of a tetrahedron results to another tetrahedron and thus the symmetrical group of both the dual are the same.

Lemma20: Let $v_i\ \epsilon\ V$ where $V$the set of the vertices of a tetrahedron is. Then the orbit of v_i is of length 4.

Proof: By ***definition 6***, we get

$$Orb_G(v_i) = V, and\ since \mid V \mid = 4,$$

$$then \mid Orb_G(v_i) \mid = 4.$$

Hence the orbits are of length 4.

Example2: Take $\boldsymbol{v_1 \epsilon V}$ then its corresponding orbit in G is given by

$$Orb_G(v_1) = \{v_1, v_2, v_3, v_4\} = V$$

i.e.$\mid Orb_G(v_1) \mid = 4.$

Lemma21: The order of a stabilizer of $v_i \epsilon V$ in G is 3.

Proof: Let X be the tetrahedron. The non-identity element of G fixing the point $\boldsymbol{v_i \epsilon V}$is $\boldsymbol{g_j}$ i.e. rotations through the point. Any rotational axis going through a vertex and the centre of gravity of X must go through the centre of the opposite face which is an equilateral. Thus, the angle of rotation that leaves X invariant is$\left\{\frac{2\pi}{3}\right\}$. Consequently, the order of $\boldsymbol{g_j}$ and hence the stabilizer is 3.

Example 3: Take $v_1 \epsilon V$then using ***Definition 8*** we obtain the following result:-

$$\boldsymbol{Stab_G(v_1) = \{I, (234), (243)\}}$$

$$\boldsymbol{i.e. |Stab_G(v_1)| = 3}$$

Theorem22: The action of G on the set of vertices V is transitive.

Proof: From the ***Lemma 21*** and ***Definition 7*** above,$\boldsymbol{| fix(v_i)| = 3}$. By the ***Theorem 12***, the number of orbits is equal to 1.

$$\text{i.e.} \boldsymbol{|Orb_G(v_i)| = 1}$$

Therefore, by ***Definition 10***, the action of G on V is transitive.

Theorem 23: The action of G on the set of vertices V is Faithful.

Proof: We observe that only $g_1(v_i) = v_i,\ g_1 = I$ fixes any of the vertices while the rest of the elements of G maps the vertices to different points. Then by ***Definition 13*** above the action of G on V is indeed faithful.

Theorem24: The action of G on the set of vertices V is not regular.

Proof: Using ***Definition 10*** and ***Lemma 21*** the action of G on V is transitive but not regular since

$$\boldsymbol{|Stab_G(v_i)| \neq 3}$$

Theorem25: The action of G on the set of vertices V is primitive.

Proof: For any $v_i, i = 1,2,3,4$ the isotropy subgroup$\boldsymbol{G_{v_i}}$ is of order 3 and cyclic. Since the order of $\boldsymbol{G_{v_i}}$is prime, the subgroup is maximal. Hence, by ***Theorem 11***, the action is primitive.

Example 4: Let $\boldsymbol{G \cong A_4}$and$\boldsymbol{V = \{v_1, v_2, v_3, v_4\}}$. Then

$$\boldsymbol{G_{v_2} = \{I, (134), (143)\}}$$

This is a cyclic group since $\boldsymbol{G_{v_2} = \langle(134)\rangle}$ of order three. Hence maximal.

Theorem26: The action of G on V has sub-degrees $1^{(1)}$and$1^{(3)}$. Thus the rank is 2.

Proof: By ***Lemma 21***, the order of the isotropy subgroup $\boldsymbol{G_{v_i}}$ on V is 3. The action of $\boldsymbol{G_{v_i}}$has 2 orbits namely;

- The trivial$\boldsymbol{\Delta_i}$
- The non-trivial$\boldsymbol{\Lambda_i}$

Where$\boldsymbol{\Delta_i}$ denotes the element of **V** being acted upon by $\boldsymbol{G_{v_i}}$and $\boldsymbol{\Lambda_i}$ denotes the set $\{\boldsymbol{V \backslash v_i}\}$with v_i being the element of **V** being acted upon. We observe that $\boldsymbol{|\Delta_i| = 1}and\boldsymbol{|\Lambda_i| = 3}$.

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