Properties of the Rotational Symmetry Group of a Tetrahedron Acting on Its Vertices

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Abstract: - Platonic solids are 3-dimensional solids that have been a subject of interest to mathematician for years due to their natural unique and regular properties. Euclid (300BC) showed that there are only 5 such solids; tetrahedron, cube, octahedron, dodecahedron and icosahedron. Robert moon in the 20th century expanded the link between the platonic solids and the physical world to the electron shell model in chemistry. To delve into this area, this study deals with the action of rotational symmetry groups of the platonic solids on their respective vertices. The action is showed to be transitive, primitive and semi-regular. The sub-degrees of the action of rotational symmetry group of a tetrahedron on its vertices are \(1^{(3)}\text{and}1^{(3)}\). The rank is 2.

AMS Subject Classification: 20B05

Key Words: Rank, Sub-degrees, Transitivity.

I. INTRODUCTION

The Platonic solids in a 3-dimensional space are the convex regular polyhedrons which are constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. There are only 5 solids that meet this criteria; tetrahedron, cube, icosahedron, dodecahedron, and icosahedron. In this paper, our interest is reserved to the tetrahedron, cube, icosahedron, dodecahedron, and icosahedron. Robert moon in the 20th century expanded the link between the platonic solids and the physical world to the electron shell model in chemistry. To delve into this area, this study deals with the action of rotational symmetry groups of the platonic solids on their respective vertices. The action is showed to be transitive, primitive and semi-regular. The sub-degrees of the action of rotational symmetry group of a tetrahedron on its vertices are \(1^{(3)}\text{and}1^{(3)}\). The rank is 2.

II. PRELIMINARY NOTES

2.1 Regular Solids

Definition 1: [4] A Platonic solid is a convex regular polyhedron with congruent faces, edges and vertices.

Definition 2: A dual of a platonic solid is a similar solid formed by forming vertices above the centres of each face and then connecting them with the adjacent vertices.

2.2 Permutation Groups

Definition 3: Permutations of a set \(X\) can be defined as a bijection from the set \(X\) onto itself. All permutations of a set \(Gx = x\)

Theorem 11: [2] Let \(x \in X\), \(|X| > 1\). A transitive group on \(X\) is primitive if and only if \(G_x\) is a maximal group of \(G\).

Theorem 12: (Cauchy-Frobenius theorem) Let \(G\) act on a non-empty set. Then the number of orbits of \(X\) is equal to

\[
\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|
\]

Where \(\text{fix}(g)\) is the number of elements of \(X\) fixed by \(g \in G\) is given by;

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tetrahedron described above is isomorphic to the
Remark 3.1: The group of rotational symmetries of a tetrahedron described above is isomorphic to the Alternating group $A_4$. Therefore;

$$\text{fix}(g) = \{geG | gx = x\}$$

Definition 13: [1] Let $G$ be a group and $g \in G$. Let $X$ be a set with $x \in X$. Then the action of $G$ on $X$ is said to be faithful if $gx = x$; for all $x \epsilon X$ implies $g = x$ the identity element in $G$.

Definition 14: [5] Let a group $G$ act transitively on a non-empty set $X$. The orbits of the isotropy $G_x$ of a point $x \epsilon X$ is called suborbits of $G$ on $X$. The number $R(G)$ of these suborbits is known as the rank of $G$ on $X$ and the lengths of the suborbits are called the sub-degrees of $G$ on $X$.

Definition 15: Let an action of a group $G$ on a set $X$ be transitive. Then the action is said to be regular if it is also semi-regular. I.e. if

$$\text{Stab}_G(x) = \text{the identity in } G.$$  

Definition 16: [6] Suppose that the action of a group $G$ on a non-empty finite set $X$ is transitive. Then a subset $\sigma$ of $X$ is referred to as a block of the action if, for each $g \in G$, either $g \sigma = \sigma$ or $g \sigma \cap \sigma = \phi$. In particular, $\phi$, $X$ and all $1$ -element subsets of $X$ are obvious blocks and they are referred to as trivial blocks. If the action of $G$ on the set $X$ has only trivial blocks then action is primitive and imprimitive otherwise.

III. MAIN RESULTS

3.1 Group of Rotational Symmetries of a Tetrahedron

A tetrahedron has 4 vertices, 6 edges and 4 faces. Suppose we let the solid be situated with its centre at the origin in $R^3$. The group of rotational symmetries in $R^3$ leaving the tetrahedron invariant will be denoted by $G$. Its elements are

- The identity, $I$,
- Two rotations through $120^0$ and $240^0$about each of the four axes joining vertices with centres of opposite faces,
- One rotation through angle $180^0$ about each of 3joining the midpoints of opposites edges. We note this group isomorphic to $A_4$.

These are as illustrated in the figure below

![Figure 1: Rotational Symmetries of a Tetrahedron](image)

**Remark 3.1:** The group of rotational symmetries of a tetrahedron described above is isomorphic to the Alternating group $A_4$. Therefore;

$$G = \{g_1 = I, g_2 = (123), g_3 = (124), g_4 = (132), g_5 = (134), g_6 = (143), g_7 = (124), g_8 = (234), g_9 = (243), g_{10} = (12)(34), g_{11} = (13)(24), g_{12} = (14)(23)\}$$

**Theorem 17:** Let $X = \{v_i, v_2, v_3, v_4\}$ be the set of vertices of a tetrahedron where $v_i$ represents each vertex of the solid. Let $G$ be the group of rotational symmetries of the solid as described above. Then we define the action of $G$ on $Y$ by

$$g_i(v_j) = v_k$$

Where $g_i \epsilon G, i = 1, 2, ..., 12$ and $v_j, v_k \epsilon Y$ with $j, k = 1, 2, 3, 4$.

**Proof:**

1. From Remark 3.1 above, $g_1$ is the identity in $G$, and hence

$$g_i(v_j) = (v_j), \forall v_j \epsilon Y$$

2. Let $g_i, g_j \epsilon G$, then $(g_i g_j) v_k = g_i (g_j v_k)$

**Example 1:**

1. Taking $g_1 = I$ and $v_4$ then $g_1(v_4) = v_4$
2. Taking $g_2 = (123)$, $g_3 = (124) \epsilon G$ we have,

$$(g_2 g_3) v_4 = ((123)124)v_4$$

$$= (13)(24)v_4$$

$$= v_2$$

On the other hand,

$$g_2(g_3 v_4) = (123)((124)v_4)$$

$$= (123)v_4$$

$$= v_2$$

**Conclusion 1:** Hence the action is well-defined.

**Lemma 18:** [4] All dual polyhedrons form the similar symmetrical groups.

**Proof:** Let $P$ be a platonic solid. By placing points on the centre of every face and then connecting the points to points in neighbouring faces of the original polyhedron will form the dual of the original platonic solid.


**Proof:** From Definition 2, Definition 3 and Lemma 18 we observe that the formation of the dual of a tetrahedron results to another tetrahedron and thus the symmetrical group of both the dual are the same.

**Lemma 20:** Let $v_i \epsilon V$ where $V$ the set of the vertices of a tetrahedron is. Then the orbit of $v_i$ is of length 4.
**Theorem 24:** Therefore, by transitive.

**Example 2:** Take $v_1 \in V$ then its corresponding orbit in $G$ is given by

$$\text{Orb}_G(v_1) = \{v_1, v_2, v_3, v_4\} = V$$

i.e. $|\text{Orb}_G(v_1)| = 4$.

**Lemma 21:** The order of a stabilizer of $v_i \in V$ in $G$ is 3.

**Proof:** Let $X$ be the tetrahedron. The non-identity element of $G$ fixing the point $v_i \in V$ is $g_j$ i.e. rotations through the point. Any rotational axis going through a vertex and the centre of gravity of $X$ must go through the centre of the opposite face which is an equilateral. Thus, the angle of rotation that leaves $X$ invariant is $\{\frac{2\pi}{3}\}$. Consequently, the order of $g_j$ and hence the stabilizer is 3.

**Example 3:** Take $v_i \in V$ then using **Definition 8** we obtain the following result:-

$$\text{Stab}_G(v_1) = \{(1, 234), (243)\}$$

i.e. $|\text{Stab}_G(v_1)| = 3$

**Theorem 22:** The action of $G$ on the set of vertices $V$ is transitive.

**Proof:** From the **Lemma 21** and **Definition 7** above, $|\text{fix}(v_j)| = 3$. By the **Theorem 12**, the number of orbits is equal to 1.

i.e. $|\text{Orb}_G(v_j)| = 1$

Therefore, by **Definition 10**, the action of $G$ on $V$ is transitive.

**Theorem 23:** The action of $G$ on the set of vertices $V$ is Faithful.

**Proof:** We observe that only $g_1(v_i) = v_i$, $g_1 = I$ fixes any of the vertices while the rest of the elements of $G$ maps the vertices to different points. Then by **Definition 13** above the action of $G$ on $V$ is indeed faithful.

**Theorem 24:** The action of $G$ on the set of vertices $V$ is not regular.

**Proof:** Using **Definition 10** and **Lemma 21** the action of $G$ on $V$ is transitive but not regular since $|\text{Stab}_G(v_i)| \neq 3$

**Theorem 25:** The action of $G$ on the set of vertices $V$ is primitive.

**Proof:** For any $v_i, i = 1, 2, 3, 4$ the isometry subgroup $G_{v_i}$ is of order 3 and cyclic. Since the order of $G_{v_i}$ is prime, the subgroup is maximal. Hence, by **Theorem 11**, the action is primitive.

**Example 4:** Let $G \cong A_4$ and $V = \{v_1, v_2, v_3, v_4\}$. Then

$$G_{v_2} = \{(1, 134), (143)\}$$

This is a cyclic group since $G_{v_2} = \langle (134) \rangle$ of order three. Hence maximal.

**Theorem 26:** The action of $G$ on $V$ has sub-degrees $1^{(1)}$ and $1^{(3)}$. Thus the rank is 2.

**Proof:** By **Lemma 21**, the order of the isometry subgroup $G_{v_i}$ on $V$ is 3. The action of $G_{v_i}$ has 2 orbits namely;

- The trivial $A_i$
- The non-trivial $A_i$

Where $A_i$ denotes the element of $V$ being acted upon by $G_{v_i}$ and $A_i$ denotes the set $|V \setminus v_i|$ with $v_i$ being the element of $V$ being acted upon. We observe that $|A_i| = 1$ and $|A_i| = 3$.

**REFERENCES**


