Let H be a transitive permutation group acting on a set Y. Then H acts on Y × Y by h(x,y) = (hx,hy), h ∈ H, x,y ∈ Y. If O ⊆ Y × Y is a H-orbit, then for a fixed x ∈ Y, ∆ = {y ∈ Y: (x, y) ∈ O} is a Hx-orbit. On the other hand, if ∆ ⊆ Y is a Hx-orbit, then O = {(hx,hy)|h ∈ H, y ∈ ∆} is a H-orbit on Y × Y. We say ∆ corresponds to O. The Hx-orbits on Y are called suborbits of H. The rank of H in this case is r.

Theorem 2.5 (Benson and Grove [1])

Let G be a finite group acting on a finite set X, then the number of G-orbits is

\[ \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|, \]

where \( \text{Fix}(h) = \{ y \in X : hx = y \} \).

Theorem 2.6 (Sims [5])

Let H be transitive on a set Y and let \( \text{Hy} \) be the stabilizer in H of a point \( y \in Y \). The orbits \( \Delta_0 = \{ y \}, \Delta_1, \ldots, \Delta_{r-1} \) of \( \text{Hy} \) on Y are called the suborbits of H. The rank of H in this case is r.

Definition 2.4

The sizes \( n_i = |\Delta_i|, (i = 0, 1, 2, \ldots, r - 1) \) often called the ‘lengths’ of the suborbits are known as subdegrees of H.

Definition 2.2

A permutation group G acting on a set X is said to be transitive if for all \( x, y \in X \) there exists an element \( g \in G \) such that \( gx = y \). Alternatively, if the action of the group G on the set X has one orbit, then it is said that G acts transitively on X.

II. NOTATIONS AND PRELIMINARY RESULTS

Notation 2.1

Throughout this paper, \( \Gamma \) denotes the symmetry group of a tetrahedron, \( \Gamma \) is the suborbital graph corresponding to the suborbit \( \Delta \) and \( O \) is the suborbital of G on X×X.

III. MAIN RESULTS

Figure 1: A tetrahedron with labelled edges
3.1 Group of Symmetries of a Tetrahedron

Suppose that a tetrahedron is situated such that its center is at the origin in \( \mathbb{R}^3 \). The subgroup of the rotations in \( \mathbb{R}^3 \) which leave the tetrahedron invariant is denoted by \( \Delta \). The elements of \( \Delta \) include:

I. Two rotations through angles of 120° and 240° about each of the four axes joining the vertices to the center of opposite faces.

II. A rotation through the 180° angle about each of the three axes joining the midpoints of opposite edges.

III. The identity.

Thus, \( |\Delta| = 4.2 + 3.1 + 1 = 12 \). The group of symmetries of a tetrahedron is isomorphic to \( A_4 \) (See [1] for more details).

3.2 Cycle Type, Stabilizer and Transitivity of \( A_4 \) acting on the edges of a Tetrahedron

<table>
<thead>
<tr>
<th>Rotations</th>
<th>Number of Permutations</th>
<th>Cycle Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotations about each of the four axes joining the vertices with centers of opposite faces through 120° and 240°</td>
<td>8</td>
<td>(0,0,2,0,0,0)</td>
</tr>
<tr>
<td>Rotation through the 180° angle about each of the three axes joining the midpoints of opposite edges</td>
<td>3</td>
<td>(2,2,0,0,0,0)</td>
</tr>
<tr>
<td>The Identity</td>
<td>1</td>
<td>(6,0,0,0,0,0)</td>
</tr>
</tbody>
</table>

The identity rotation stabilizes all the edges. Moreover, the rotation through 180° angle about each of the three axes joining the midpoints of opposite edges stabilizes opposite edges of a tetrahedron. Consequently, the stabilizer of the action of \( A_4 \) acting on the edges of a tetrahedron is \( \{1, 180°\} \), which is a cyclic group of order 2.

From Table 1, we can compute the transitivity by first obtaining \( \text{Fix}(g) \) for each \( g \in A_4 \), then using the Cauchy-Frobenius lemma, we find the number of orbits when \( A_4 \) acts on the edges of the tetrahedron. The equation is given by

\[
\frac{1}{12} (8.0 + 3.2 + 1.6) = 1
\]

Therefore, from Definition 2.2, \( A_4 \) acts transitively on the edges of a tetrahedron.

3.3 Rank and Subdegrees of \( \Delta \) acting on the edges of a Tetrahedron

Using Definition 2.3 and the Cauchy-Frobenius lemma we can compute the rank of the action of the symmetry group on the edges of a tetrahedron whereby:

\[
\text{Fix}(g) = \{x \in X : gx = x\} = \frac{1}{|\Delta|} \sum_{g \in \Delta} |\text{Fix}(g)|, \quad \text{where}
\]

\[
\text{Fix}(g) = \{x \in X : gx = x\} \quad \text{and} \quad X \text{ is the symmetry group of a tetrahedron acting on its edges, } \Delta. \quad \text{Hence, we have the rank as:}
\]

\[
\frac{8}{12} (6 + 2) = \frac{8}{2} = 4
\]

Where 6 is the number of edges fixed by the identity and 2 is the number of edges fixed by the rotation of 180° angle about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer and Definition 2.4, we can obtain the corresponding subdegrees of the action of \( \Delta \) on the edges of a tetrahedron. From Figure 1 with the axis of rotation joining the midpoints of edges labelled 1 and 5, under the 180° angle rotation we obtain the following permutation \( (1) (5) (3) (4) \) whereby, the subdegrees are 1, 1, 2, 2 to the corresponding suborbits \( \Delta_0 = \{1\}, \Delta_1 = \{5\}, \Delta_2 = \{3, 4\}, \Delta_3 = \{2, 4\} \) respectively.

We can change the axis to start from other opposite edges, for example 3 and 4, but the subdegrees will remain the same since we already showed the action is transitive with one orbit.

3.4 Suborbital graphs of \( \Delta \) acting on the edges of a Tetrahedron

After computing the subdegrees, we proceed to analyze the suborbits whereby, if \( G \) acts on a set \( X \) then \( G \) acts on \( X \times X = \{(a, b) \mid a, b \in X\} \) by \( g(a, b) = (ga, gb) \) and \( \text{Orb}_G(a, b) \) is the orbital or suborbit of \( G \) containing \( (a, b) \), denoted by \( O_i \). In our case, \( G \) denotes the symmetry group of a tetrahedron acting on its edges and \( X \) is the set of edges, as labeled in Figure 1, \( \{1, 2, 3, 4, 5, 6\} \). For example, for 180° \( \in G \) such that 180° \( (1, 2) = \{(1, 6), (5, 2), (5, 6)\} \). The rank is equal to the total number of suborbital graphs, so we need to find 4 orbitals. In this case we have that:

\[
\text{Orb}_G(1, 1) = O_0 = \{1, 1\} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}
\]

\[
\text{Orb}_G(1, 2) = O_1 = \{(1, 2), (6, 3), (4, 5), (2, 3), (3, 1), (2, 4), (4, 1), (6, 4)\}
\]

\[
\text{Orb}_G(2, 1) = O_2 = \{(1, 2), (6, 3), (4, 5), (2, 3), (3, 1), (2, 4), (4, 1), (6, 4)\}
\]

\[
\text{Orb}_G(1, 5) = O_3 = \{(1, 5), (6, 2), (4, 3), (2, 6), (3, 4), (2, 6), (4, 3), (6, 2), (3, 4), (1, 5)\}
\]

Next we construct the corresponding suborbital graphs. The suborbital graph corresponding to the suborbit \( \Delta_0 \) is the trivial graph, \( \Gamma_0 \), and so we ignore it.
$\Gamma_1$: 

$\Gamma_2$: 

3.5 Properties of the Suborbital Graphs

The graphs $\Gamma_1$ and $\Gamma_2$ are directed, connected and their girth is 3. The graph $\Gamma_3$ is undirected, disconnected and has 3 connected components.

From Theorem 2.6, we find that the action of $\mathbb{I}$ is imprimitive, as $\Gamma_3$ is disconnected.

We can also confirm the number of self-paired graphs, using Theorem 2.7, such that for the equation $\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g^2)|$, where $G$ is the symmetry group acting on the edges of a tetrahedron and using Table 1, we have:

$$\frac{1}{12} \left(6 \cdot 3 + 6 \cdot 1\right) = \frac{24}{12} = 2$$

Where the 180° angle rotation about the center of opposite edges becomes the identity rotation when squared hence, fixes 6 edges with 3 permutations. Also, the identity rotation fixes 6 edges with 1 permutation. We have the following self-paired graphs; $\Gamma_0$ and $\Gamma_3$.

IV. CONCLUSION

The tetrahedron is a unique solid full of interesting mathematical properties for research and so is the reason why it is the focus of this research.

REFERENCES