Kamal Transform of Bessel's Functions

Sudhanshu Aggarwal

Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur-273402, U.P., India

Abstract: **In the modern time, Bessel's functions appear in solving many problems of sciences and engineering together with many equations such as Schrodinger equation, heat equation, wave equation, Laplace equation, Helmholtz equation in cylindrical or spherical coordinates. In this paper, we determine Kamal transform of Bessel's functions. Some applications of Kamal transform of Bessel's functions for evaluating the integral, which contain Bessel's functions, are given.**

Keywords: Kamal transform, Convolution theorem, Inverse Kamal transform, Bessel function.

I. INTRODUCTION

Bessel's functions have many applications [3] to solve the problems of mathematical physics, acoustics, engineering and sciences such as heat transfer, fluid mechanics, vibrations, stress analysis, hydrodynamics, flux distribution in a nuclear reactor etc.

Bessel's function of order n, where $n\epsilon N$ is given by [1-5,10]

$$J_n(t)=\frac{t^n}{2^n n!}\left[1-\frac{t^2}{2.(2n+2)}+\frac{t^4}{2.4.(2n+2)(2n+4)}-\frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\cdots\right]\ldots\ldots(1)$$

In particular, when $n=0$, we have Bessel's function of zero order and it is denoted by $J_0(t)$and it is given by the infinite power series

$$J_0(t)=1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}-\frac{t^6}{2^2.4^2.6^2}+\cdots..(2)$$

For $n=1$, we have Bessel's function of order one and it is denoted by $J_1(t)$and it is given by

$$J_1(t)=\frac{t}{2}-\frac{t^3}{2^2.4}+\frac{t^5}{2^2.4^2.6}-\frac{t^7}{2^2.4^2.6^2.8}+..(3)$$

Equation (3) can be written as

$$J_1(t)=\frac{t}{2}-\frac{t^3}{2^3.2!}+\frac{t^5}{2^5.2!.3!}-\frac{t^7}{2^7.3!.4!}+\cdots.(4)$$

For $n=2$, we have Bessel's function of order two and it is denoted by $J_2(t)$and it is given by

$$J_2(t)=\frac{t^2}{2.4}-\frac{t^4}{2^2.4.6}+\frac{t^6}{2^2.4^2.6.8}-\frac{t^8}{2^2.4^2.6^2.8.10}+\cdots..(5)$$

The Kamal transform of the function $F(t)$is defined as [6]:

$$K\{F(t)\}=\int_0^\infty F(t)e^{\frac{-t}{v}}dt$$
$$=G(v),t\geq 0,k_1\leq v\leq k_2\ldots\ldots..(6)$$

Where K is Kamal transform operator.

The Kamal transform of the function $F(t)$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Kamal tansform of the function $F(t)$.

Abdelilah and Hassan [7] used Kamal transform for solving partial differential equations. The convolution for Kamal and Mahgoub transforms was given by Fadhil [8]. Taha et al. [9] defined the dualities between Kamal and Mahgoub integral transforms and these authors also gave some famous integral transforms. Aggarwal et al. [12] discussed a new application of Kamal transform for solving linear Volterra integral equations.

The object of the present study is to determine Kamal transform of Bessel's functions and explain the advantage of Kamal transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

II. LINEARITY PROPERTY OF KAMAL TRANSFORM

$$K\{aF(t)+bG(t)\}=aK\{F(t)\}+bK\{G(t)\}$$

Where a,b are arbitrary constants.

III. KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 8]

S.N.	$F(t)$	$K\{F(t)\}=G(v)$
1.	1	v
2.	t	v^2
3.	t^2	$2!\,v^3$
4.	$t^n, n\epsilon N$	$n!\,v^{n+1}$
5.	$t^n, n>-1$	$\Gamma(n+1)v^{n+1}$
6.	e^{at}	$\frac{v}{1-av}$
7.	$sinat$	$\frac{av^2}{1+a^2v^2}$
8.	$cosat$	$\frac{v}{1+a^2v^2}$
9.	$sinhat$	$\frac{av^2}{1-a^2v^2}$
10.	$coshat$	$\frac{v}{1-a^2v^2}$

IV. CHANGE OF SCALE PROPERTY OF KAMAL TRANSFORM

If $K\{F(t)\} = G(v)$ then

$$K\{F(at)\} = \int_0^\infty F(at)e^{\frac{-t}{v}}dt \ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

Put $at = p \Rightarrow adt = dp$ in equation (7), we have

$K\{F(at)\} = \frac{1}{a}\int_0^\infty F(p)e^{\frac{-p}{av}}dp = \frac{1}{a}G(av)$

Thus, if $K\{F(t)\} = G(v)$ then

$$K\{F(at)\} = \frac{1}{a}G(av) \ldots\ldots\ldots\ldots\ldots.(8)$$

V. KAMAL TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $F(t)$ [6, 8, 9]

If $K\{F(t)\} = G(v)$ then

a) $K\{F'(t)\} = \frac{1}{v}G(v) - F(0)$
b) $K\{F''(t)\} = \frac{1}{v^2}G(v) - \frac{1}{v}F(0) - F'(0)$
c) $K\{F^{(n)}(t)\} = \frac{1}{v^n}G(v) - \frac{1}{v^{n-1}}F(0) - \frac{1}{v^{n-2}}F'(0) \ldots\ldots - F^{(n-1)}(0)$

VI. CONVOLUTION OF TWO FUNCTIONS [11]

Convolution of two functions $F(t)$ and $H(t)$ is denoted by $F(t) * H(t)$ and it is defined by

$$F(t) * H(t) = F * H = \int_0^t F(x)H(t-x)dx$$
$$= \int_0^t H(x)F(t-x)dx$$

VII. CONVOLUTION THEOREM FOR KAMAL TRANSFORMS [8]

If $K\{F(t)\} = G(v)$ and $K\{H(t)\} = I(v)$ then

$$K\{F(t) * H(t)\} = K\{F(t)\}K\{H(t)\} = G(v)I(v)$$

VIII. INVERSE KAMAL TRANSFORM [12]

If $K\{F(t)\} = G(v)$ then $F(t)$ is called the inverse Kamal transform of $G(v)$ and mathematically it is defined as

$F(t) = K^{-1}\{G(v)\}$

where K^{-1} is the inverse Kamal transform operator.

IX. INVERSE KAMAL TRANSFORM OF SOME ELEMENTARY FUNCTIONS [12]

S.N.	$G(v)$	$F(t) = K^{-1}\{G(v)\}$
1.	v	1
2.	v^2	t
3.	v^3	$\frac{t^2}{2!}$
4.	$v^{n+1}, n\epsilon N$	$\frac{t^n}{n!}$
5.	$v^{n+1}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{v}{1-av}$	e^{at}
7.	$\frac{v^2}{1+a^2v^2}$	$\frac{sinat}{a}$
8.	$\frac{v}{1+a^2v^2}$	$cosat$
9.	$\frac{v^2}{1-a^2v^2}$	$\frac{sinhat}{a}$
10.	$\frac{v}{1-a^2v^2}$	$coshat$

X. RELATION BETWEEN $J_0(t)$ AND $J_1(t)$[5, 10]

$$\frac{d}{dt}J_0(t) = -J_1(t) \ldots\ldots\ldots.(9)$$

XI. RELATION BETWEEN $J_0(t)$ AND $J_2(t)$[10]

$$J_2(t) = J_0(t) + 2J_0''(t) \ldots\ldots\ldots(10)$$

XII. KAMAL TRANSFORM OF BESSEL'S FUNCTIONS

a) Kamal transform of $J_0(t)$:

Taking Kamal transform of equation(2), both sides, we have

$K\{J_0(t)\} = K\{1\} - \frac{1}{2^2}K\{t^2\} \quad + \frac{1}{2^2.4^2}K\{t^4\} - \frac{1}{2^2.4^2.6^2}K\{t^6\} + \cdots..$

$$= v - \frac{1}{2^2}(2!\,v^3) + \frac{1}{2^2.4^2}(4!\,v^5) - \frac{1}{2^2.4^2.6^2}(6!\,v^7) + \cdots\ldots\ldots.$$

$$= v\left[1 - \frac{1}{2}(v^2) + \frac{1.3}{2.4}(v^2)^2 - \frac{1.3.5}{2.4.6}(v^2)^3 + \cdots\ldots\ldots\ldots.\right]$$

$$= v(1+v^2)^{-1/2} = \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots..(11)$$

b) Kamal transform of $J_1(t)$:

Taking Kamal transform of equation (9), both sides, we have

$$K\{J_1(t)\} = -K\{J_0'(t)\} \ldots\ldots\ldots\ldots\ldots.(12)$$

Now applying the property, Kamal transform of derivative of the function on equation(12), we have

$$K\{J_1(t)\} = -\left[\frac{1}{v}K\{J_0(t)\} - J_0(0)\right] \ldots.(13)$$

Using equation (2) and equation (11) in equation (13), we have

$$K\{J_1(t)\} = -\left[\frac{1}{v}.\frac{v}{\sqrt{(1+v^2)}} - 1\right]$$

$$K\{J_1(t)\} = 1 - \frac{1}{\sqrt{(1+v^2)}} \dots\dots\dots(14)$$

c) Kamal transform of $J_2(t)$:

Taking Kamal transform of equation (10), both sides, we have

$$K\{J_2(t)\} = K\{J_0(t)\} + 2K\{J_0''(t)\} \dots\dots(15)$$

Now applying the property, Kamal transform of derivative of the function and using equation (11) in equation (15), we have

$$K\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + 2\left[\frac{1}{v^2}K\{J_0(t)\} - \frac{1}{v}J_0(0) - J_0'(0)\right] \dots\dots(16)$$

Using equation (2), equation (9) and equation (11) in equation (13), we have

$$K\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + 2\left[\frac{1}{v^2}.\frac{v}{\sqrt{(1+v^2)}} - \frac{1}{v} + J_1(0)\right] \dots\dots\dots\dots\dots\dots(17)$$

Using equation (3) in equation (17), we have

$$K\{J_2(t)\} = \frac{v}{\sqrt{(1+v^2)}} + \frac{2}{v\sqrt{(1+v^2)}} - \frac{2}{v}$$

$$= \frac{v^2 + 2 - 2\sqrt{(1+v^2)}}{\boldsymbol{v}\sqrt{(1+v^2)}} \dots\dots\dots(18)$$

d) Kamal transform of $J_0(at)$:

From equation (11), Kamal transform of $J_0(t)$ is given by

$$K\{J_0(t)\} = \frac{v}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Kamal transform, we have

$$K\{J_0(at)\} = \frac{1}{a}\left[\frac{av}{\sqrt{(1+(av)^2)}}\right]$$

$$= \left[\frac{v}{\sqrt{(1+a^2v^2)}}\right] \dots\dots\dots\dots\dots(19)$$

e) Kamal transform of $J_1(at)$:

From equation (14), Kamal transform of $J_1(t)$ is given by

$$K\{J_1(t)\} = 1 - \frac{1}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Kamal transform, we have

$$K\{J_1(at)\} = \frac{1}{a}\left[1 - \frac{1}{\sqrt{(1+(av)^2)}}\right]$$

$$= \frac{1}{a}\left[1 - \frac{1}{\sqrt{(1+a^2v^2)}}\right] \dots\dots\dots\dots\dots(20)$$

f) Kamal transform of $J_2(at)$:

From equation (18), Kamal transform of $J_2(t)$ is given by

$$K\{J_2(t)\} = \frac{v^2 + 2 - 2\sqrt{(1+v^2)}}{\boldsymbol{v}\sqrt{(1+v^2)}}$$

Now applying change of scale property of Kamal transform, we have

$$K\{J_2(at)\} = \frac{1}{a}\left[\frac{(av)^2 + 2 - 2\sqrt{(1+(av)^2)}}{(av)\sqrt{(1+(av)^2)}}\right]$$

$$= \frac{1}{a^2}\left[\frac{a^2v^2 + 2 - 2\sqrt{(1+a^2v^2)}}{v\sqrt{(1+a^2v^2)}}\right] \dots\dots\dots\dots(21)$$

XIII. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Kamal transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

Application: 1 Evaluate the integral

$$I(t) = \int_0^t J_0(u)J_0(t-u)du \dots\dots\dots(22)$$

Applying the Kamal transform to both sides of (22), we have

$$K\{I(t)\} = K\left\{\int_0^t J_0(u)J_0(t-u)du\right\} \dots.(23)$$

Using convolution theorem of Kamal transform on (23), we have

$$K\{I(t)\} = K\{J_0(t)\}K\{J_0(t)\}$$

$$= \frac{v}{\sqrt{(1+v^2)}}.\frac{v}{\sqrt{(1+v^2)}} = \frac{v^2}{1+v^2} \dots(24)$$

Operating inverse Kamal transform on both sides of (24), we have

$$I(t) = K^{-1}\left\{\frac{v^2}{1+v^2}\right\} = sint \dots\dots (25)$$

which is the required exact solution of (22).

Application:2 Evaluate the integral

$$I(t) = \int_0^t J_0(u)J_1(t-u)du \dots\dots\dots.(26)$$

Applying the Kamal transform to both sides of (26), we have

$$K\{I(t)\} = K\left\{\int_0^t J_0(u)J_1(t-u)du\right\} \dots.(27)$$

Using convolution theorem of Kamal transform on (27), we have

$$K\{I(t)\} = K\{J_0(t)\}K\{J_1(t)\}$$

$$= \frac{v}{\sqrt{(1+v^2)}}.\left[1 - \frac{1}{\sqrt{(1+v^2)}}\right]$$

$$= \frac{v}{\sqrt{(1+v^2)}} - \frac{v}{1+v^2} \dots\dots (28)$$

Operating inverse Kamal transform on both sides of (28), we have

$$I(t) = K^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\} - K^{-1}\left\{\frac{v}{1+v^2}\right\}$$

$$= J_0(t) - cost \dots\dots\dots\dots\dots.(29)$$

which is the required exact solution of (26).

Application:3 Evaluate the integral

$$I(t) = \int_0^t J_1(t-u)du \dots\dots\dots.(30)$$

Applying the Kamal transform to both sides of (30), we have

$$K\{I(t)\} = K\left\{\int_0^t J_1(t-u)du\right\} \dots.(31)$$

Using convolution theorem of Kamal transform on (31), we have

$$K\{I(t)\} = K\{1\}K\{J_1(t)\}$$

$$= v.\left[1 - \frac{1}{\sqrt{(1+v^2)}}\right]$$

$$= v - \frac{v}{\sqrt{(1+v^2)}} \dots\dots (32)$$

Operating inverse Kamal transform on both sides of (32), we have

$$I(t) = K^{-1}\{v\} - K^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\}$$

$$= 1 - J_0(t) \dots\dots\dots\dots\dots.(33)$$

which is the required exact solution of (31).

XIV. CONCLUSION

In this paper, we have successfully discussed the Kamal transform of Bessel's functions. The given applications show that the advantage of Kamal transform of Bessel's functions to evaluate the integral which contain Bessel's functions.

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AUTHOR

Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory. He has published many research papers in national and international journals.