Composition of Pathway Fractional Integral Operator on \((p, q)\) -Extended Struve Function

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Abstract: In this paper we present Pathway Fractional Integral Operator involving \((p, q)\) – extended Struve function \(H_{v,p,q}(z)\), which is expressed in terms of Hadamard product of the \((p, q)\) -extended Gauss Hypergeometric function and the Fox-Wright function, \(Ψ_s(z)\). The result obtained here is also reduced to the known result of R. K. Parmar and J. Choi as special case.

Key words: \((p, q)\) -extended Struve function, Fox-Wright function, \((p, q)\) -extended Gauss Hypergeometric function, Pathway integral operator, Hadamard product.

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I. INTRODUCTION

1.1 Pathway operators

Let \(f(x) \in L(a, b), p \in C, Re(\rho) > 0, a > 0\) and let us take a pathway parameter \(\alpha < 1\). Then the pathway fractional integration operator, as an extension of (1.1), is defined and represented as follows (see [20, p. 239]):

\[
(t^p \int_0^t \frac{f(\tau)}{(1-\alpha)^{\rho}} d\tau) = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)^{\rho}} t^\rho \int_0^t f(\tau) d\tau, & \text{if } \alpha < 1; \\
\frac{1}{\Gamma(1-\alpha)^{\rho}} \int_0^t f(\tau) d\tau, & \text{if } \alpha = 1.
\end{cases}
\]

(1.1)

Where \(L(a, b)\) is the set of Lebesgue measurable functions defined on \((a, b)\).

The pathway model is introduced by Mathai [1] and studied further by Mathai and Haubold [2, 3].

1.2 Result Required

The following formula is required (see [20, eq. (12)])

\[
P_{\rho, p, q, \alpha}(t^{\beta-1}) = \frac{t^{\rho + \beta}}{\Gamma(1-\alpha)^{\rho}} \frac{\Gamma(\beta) \Gamma(\beta + \frac{p}{1-\alpha} + \frac{\alpha}{1-\alpha} + 1)}{\Gamma(\beta + \frac{p}{1-\alpha} + \frac{\alpha}{1-\alpha} + 1)}
\]

(1.2)

Where \(\alpha < 1; Re(\rho) > 0; Re(\beta) > 0\).

Recently, many authors have investigated the \((p, q)\) -variant (when \(p = q\), the \(p\)-variant) associated with a set of related higher transcendental Hypergeometric type special functions

\[
H_{v,p,q}(z) = \frac{1}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\frac{v+2n+1}{2}}}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2})}
\]

(1.6)
Also, for our present investigation, we need the concept of Hadamard product (or convolution) of two analytic functions. It can help us in decomposing a newly emerged function into two known functions. If, in particular, one of the two power series defines an entire function, then the Hadamard product series defines an entire function, too. Indeed, let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < R_f) \]

and

\[ g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (|z| < R_g) \]

be two given power series whose radii of convergence are given by \( R_f \) and \( R_g \), respectively. Then their Hadamard product is a power series defined by

\[ (f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (|z| < R_f) \]  

(1.7)

Whose radius of convergence \( R \) is

\[ \frac{1}{R} = \limsup_{n \to \infty} \left( \frac{a_n b_n}{a_n b_n} \right)^{1/n} \leq \limsup_{n \to \infty} \left( \frac{a_n}{a_n} \right)^{1/n} \limsup_{n \to \infty} \left( \frac{b_n}{b_n} \right)^{1/n} = \frac{1}{R_f R_g} \]

and \( R \geq R_f R_g \).

In this paper, we aim to compositions of the pathway integral operator involving \((p, q)\)-extended Struve function \( H_{v,p,q}(z) \). Also, we deduce those results, corresponding to the main identities, for the classical Riemann-Liouville involving the \((p, q)\)-extended Struve function \( H_{v,p,q}(z) \). Further, we show that those compositions are expressed in terms of the Hadamard product (1.5) of the \((p, q)\)-extended Gauss Hypergeometric function (see [2, p. 354, equation (7.1)])

\[ F_{p,q}(a,b;c;z) = \sum_{n=0}^{\infty} \left( \begin{array}{c} (a)_n (b)_n \end{array} \right) \frac{z^n}{(c)_n n!} \quad (|z| < R(c) > R(b) > 0) \]

(1.8)

where \( (a,b) \) is the familiar beta function (see, e.g., [7]; see also [9, Section 1.1]), and Fox-Wright function \( \psi_q(z) \) \((p, q \in \mathbb{N}_0)\) (see, e.g., [5, 4]; see also [10]):

\[ \psi_q^p \left( \begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{array} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \ldots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \ldots \Gamma(\beta_p + B_p n)} \frac{z^n}{n!} \]

(1.9)

Where the equality in the convergence condition holds true for

\[ |z| < \nu := \left( \prod_{j=1}^{p} A_j^{-A_j} \right) \left( \prod_{j=1}^{q} B_j^{-B_j} \right). \]

1.3 Pathway Integral Operator of the \((p, q)\)-extended Struve function

Here we present Pathway Integral Operator of \((p, q)\)-extended Struve function \( H_{v,p,q}(z) \) of the first kind of order \( v \).

**Theorem 1:** Let \( \alpha, \eta, \sigma, v, \omega \in \mathbb{C} \) be such that \( \min \{R(p), R(q)\} > 0, R(v) > \frac{2}{3} \).

\[ \left[ P^{(\eta, \alpha, \omega)}_{0+} \left\{ t^{\sigma-1} H_{v,p,q}(\omega t) \right\} \right](x) = \]

\[ \frac{\sqrt{\pi}}{[a(1-\alpha)]^{v+1}} \frac{\eta^v+1}{\Gamma(v+\frac{3}{2})} F_{p,q} \left[ \begin{array}{c} 1,1 \\ v+\frac{3}{2} \end{array} \right] \frac{-\omega^2 x^2}{a(1-\alpha)^2} \]

\[ \times \frac{\psi_2^p \left( \begin{array}{c} (\sigma+v+1,2),(1+\frac{\eta}{1-\alpha}) \\ 1,1 \\ \frac{3}{2} \end{array} \right) \left( \frac{\eta}{1-\alpha} + \sigma + v + 2,2 \right)}{4[a(1-\alpha)^2]} \]

(1.10)

Where \* denotes the Hadamard product in (1.7) and whose left-sided Haypergeometric fractional integral is assumed to be convergent.

**Proof**

Applying (1.2) and changing the order of integration and summation, which is valid under the given conditions here, we find

\[ \left[ P^{(\eta, \alpha, \omega)}_{0+} \left\{ t^{\sigma-1} H_{v,p,q}(\omega t) \right\} \right](x) = \]

\[ \left( w \right)^{v+1} \sum_{k=0}^{\infty} \frac{B(k+1,v+\frac{3}{2};p,q)}{(2^k B(1,v+\frac{1}{2}) k!) \left( \frac{w^2}{4} \right)^k \Gamma(v+\frac{3}{2})} \]

(1.11)
Expanding the last summation in (1.11) in terms of the Hadamard product (1.4) with the function (1.8) and (1.9), we obtain the right side of (1.10).

### 1.4 Special case

(1) The result (1.10) for Riemann–Liouville fractional integral operator defined in [20] on setting $\alpha = 0, \eta = 1$, then replacing $\eta$ by $\eta - 1$. The result in (1.10) reduces to the result in [18, eq. (2.10), pg 550].

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### REFERENCES


