Result on Common Fixed Point of Two Continuous Mappings

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***Abstract*— Let X be a closed subspace of a Hilbert Space & $T_1, T_2 : X \to X$ be continuous mappings satisfying the given condition then T_1 & T_2 have unique common fixed point in X.**

***Keywords*— Continuous map, Common Fixed Point, Banach Space, Completeness, Uniqueness.**

I. Introduction

The study of Existence & Uniqueness of Coincidence Point & Common Fixed Point of Mappings satisfying certain contractive conditions has been an interesting field of Mathematics from 1922, when Banach stated & proved his famous result (Banach Contraction Principle, **[1]**).

In recent years, several authors have been obtained common fixed point results for different classes of mappings on various spaces. In this paper, we prove unique common fixed point theorem for two continuous self mapping satisfying given condition which is generalized & motivated from the results of **[2],[3] and [4].**

II. Main result

Theorem :1

Let X be a closed subspace of Hilbert Space and $T_1, T_2 : X \to X$ be continuous mappings such that :

$$\|T_1x - T_2y\| \le a_1 \frac{\|x-y\|^p[1+\|y-T_2y\|^p]}{[1+\|x-y\|^p]} + a_2[\|x-T_1x\|^p + \|y-T_2y\|^p] + a_3[\|x-T_2y\|^p + \|y-T_1x\|^p] + a_4\|x-y\|^p$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$. Then T_1 & T_2 have unique common fixed point in X.

Proof : Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ as follows :

$$x_{2n} = T_2x_{2n-1} \quad for\ n = 1,2,3,\dots$$
$$x_{2n+1} = T_1x_{2n} \quad for\ n = 0,1,2,3,\dots$$

Suppose that $n = 2m$ for some integer m.Then

$$\|x_{n+1} - x_n\| = \|x_{2m+1} - x_{2m}\| = \|T_1x_{2m} - T_2x_{2m-1}\|$$

From the given condition we have,

$$\|x_{n+1} - x_n\|^p = \|x_{2m+1} - x_{2m}\|^p = \|T_1x_{2m} - T_2x_{2m-1}\|^p$$

$$\le a_1 \frac{\|x_{2m} - x_{2m-1}\|^p[1+\|x_{2m-1} - T_2x_{2m-1}\|^p]}{[1+\|x_{2m} - x_{2m-1}\|^p]} + a_2[\|x_{2m} - T_1x_{2m}\|^p + \|x_{2m-1} - T_2x_{2m-1}\|^p] + a_3[\|x_{2m} - T_2x_{2m-1}\|^p + \|x_{2m-1} - T_1x_{2m}\|^p] + a_4\|x_{2m} - x_{2m-1}\|^p$$

This gives

$$[1 - a_2 - 2^pa_3]\|x_{2m+1} - x_{2m}\|^p \le [a_1 + a_2 + 2^pa_3 + a_4]\|x_{2m} - x_{2m-1}\|^p$$

$$\therefore \|x_{2m+1} - x_{2m}\|^p \le p(m)\|x_{2m} - x_{2m-1}\|^p$$

where

$$p(m) = \frac{[a_1 + a_2 + 2^pa_3 + a_4]}{[1 - a_2 - 2^pa_3]} \quad for\ m = 0,1,2,3,\dots$$

Clearly

$$p(m) < 1, \forall m \ge 0 \quad as \quad a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$$

Continuing in this way one gets

$$\|x_{n+1} - x_n\|^p = \|x_{2m+1} - x_{2m}\|^p \le p(m)\|x_{2m} - x_{2m-1}\|^p \\ \vdots \\ \le p(m)^n\|x_1 - x_0\|^p$$

By Similar way one can see that above inequality is also true if n is an odd integer. Since $0 \le p(m) < 1$, the sequence $\{x_n\}$ is Cauchy sequence and therefore by completeness of X, one find $\mu \in X$ such that

$$\lim_{n\to\infty} x_n = \mu$$

Since $\{x_{2n}\}\ and\ \{x_{2n+1}\}$ are sub-sequences of $\{x_n\}$ one gets

$$\lim_{n\to\infty} x_{2n} = \mu = \lim_{n\to\infty} x_{2n+1}$$

Next since T_1 & T_2 are continuous one arrives at

$$T_1(\mu) = T_1\left(\lim_{n\to\infty} x_{2n}\right) = \lim_{n\to\infty} T_1x_{2n} = \lim_{n\to\infty} x_{2n+1} = \mu$$

$$T_2(\mu) = T_2\left(\lim_{n\to\infty} x_{2n-1}\right) = \lim_{n\to\infty} T_2x_{2n-1} = \lim_{n\to\infty} x_{2n} = \mu$$

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Hence μ is common fixed point of T_1 & T_2. Now to prove the uniqueness of common fixed point, let us take $\nu(\mu \neq \nu \in X$ to be another common fixed point of *T1 & T2.*
While $\|\mu-\nu\| \neq 0$.

Hence it follows that

$$\begin{aligned}\|\mu-\nu\|^p &= \|T_1\mu - T_2\nu\|^p \\ &\leq a_1\frac{\|\mu-\nu\|^p[1+\|\nu-T_2\nu\|^p]}{[1+\|\mu-\nu\|^p]} \\ &+ a_2[\|\mu-T_1\mu\|^p+\|\nu-T_2\nu\|^p] \\ &+ a_3[\|\mu-T_2\nu\|^p+\|\nu-T_1\mu\|^p] \\ &+ a_4\|\mu-\nu\|^p\end{aligned}$$

$$\therefore \|\mu-\nu\|^p \leq (a_1+2a_3+a_4)\|\mu-\nu\|^p$$

which is a contradiction as

$$a_1+2a_3+a_4 < a_1+2a_2+2^{p+1}a_3+a_4 < 1$$

Thus $\mu=\nu$.

Theorem : 2

Let X be a closed subspace of Hilbert Space and $T_1, T_2 : X \to X$ be continuous mappings such that :

$$\begin{aligned}\|T_1x - T_2y\| \leq\ & a_1\frac{\|x-T_1x\|^p[1+\|y-T_2y\|^p]}{[1+\|x-y\|^p]} \\ &+ a_2\frac{\|y-T_2y\|^p[1+\|x-T_1x\|^p]}{[1+\|x-y\|^p]} \\ &+ a_3[\|x-T_1x\|^p+\|y-T_2y\|^p] \\ &+ a_4[\|y-T_2y\|^p+\|x-T_1x\|^p]\end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1+a_2+2(a_3+a_4) < 1$. Then T_1 & T_2 have unique common fixed point in X.

Proof : Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ as follows :

$$x_{2n} = T_2x_{2n-1} \quad for\ n=1,2,3,\ldots$$
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Suppose that $n=2m$ for some integer m.Then

$$\|x_{n+1}-x_n\| = \|x_{2m+1}-x_{2m}\| = \|T_1x_{2m}-T_2x_{2m-1}\|$$

From the given condition we have,

$$\begin{aligned}\|x_{n+1}-x_n\|^p &= \|x_{2m+1}-x_{2m}\|^p \\ &= \|T_1x_{2m}-T_2x_{2m-1}\|^p \\ &\leq a_1\frac{\|x_{2m}-T_1x_{2m}\|^p[1+\|x_{2m-1}-T_2x_{2m-1}\|^p]}{[1+\|x_{2m}-x_{2m-1}\|^p]} \\ &+ a_2\frac{\|x_{2m-1}-T_2x_{2m-1}\|^p[1+\|x_{2m}-T_1x_{2m}\|^p]}{[1+\|x_{2m}-x_{2m-1}\|^p]} \\ &+ a_3[\|x_{2m}-T_1x_{2m}\|^p+\|x_{2m-1}-T_2x_{2m-1}\|^p] \\ &+ a_4[\|x_{2m-1}-T_2x_{2m-1}\|^p+\|x_{2m}-T_1x_{2m}\|^p]\end{aligned}$$

This gives

$$\begin{aligned}&[(1-a_1-a_3-a_4)+(1-a_1-a_2-a_3-a_4)\|x_{2m}-x_{2m-1}\|^p]\|x_{2m+1}-x_{2m}\|^p \\ &\leq [(a_2+a_3+a_4)+(a_3+a_4)\|x_{2m}-x_{2m-1}\|^p]\|x_{2m}-x_{2m-1}\|^p\end{aligned}$$

$$\therefore \|x_{2m+1}-x_{2m}\|^p \leq p(m)\|x_{2m}-x_{2m-1}\|^p$$

where

$$p(m) = \frac{(a_2+a_3+a_4)+(a_3+a_4)\|x_{2m}-x_{2m-1}\|^p}{(1-a_1-a_3-a_4)+(1-a_1-a_2-a_3-a_4)\|x_{2m}-x_{2m-1}\|^p}$$
$$for\ m=0,1,2,3,\ldots$$

Clearly

$$p(m) < 1, \forall m \geq 0 \ \ as\ a_1+a_2+2(a_3+a_4) < 1$$

Continuing in this way one gets

$$\begin{aligned}\|x_{n+1}-x_n\|^p &= \|x_{2m+1}-x_{2m}\|^p \\ &\leq p(m)\|x_{2m}-x_{2m-1}\|^p \\ &\ \vdots \\ &\leq p(m)^n\|x_1-x_0\|^p\end{aligned}$$

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While $\|\mu-\nu\| \neq 0$.

Hence it follows that

$$\begin{aligned}\|\mu-\nu\|^p &= \|T_1\mu-T_2\nu\|^p \\ &\leq a_1\frac{\|\mu-T_1\mu\|^p[1+\|\nu-T_2\nu\|^p]}{[1+\|\mu-\nu\|^p]} \\ &+ a_2\frac{\|\nu-T_2\nu\|^p[1+\|\mu-T_1\mu\|^p]}{[1+\|\mu-\nu\|^p]} \\ &+ a_3[\|\mu-T_1\mu\|^p+\|\nu-T_2\nu\|^p] \\ &+ a_4[\|\nu-T_2\nu\|^p+\|\mu-T_1\mu\|^p]\end{aligned}$$
$$\therefore \|\mu-\nu\|^p \leq 0$$

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which is a contradiction .

Thus $\mu = \nu$.

III. CONCLUSIONS

The method adopted in the proof of common fixed point theorems reveal that yet there are various directions in which the Banach's fixed point theorem can be refined and extended.

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