Evaluation of Natural Frequencies of Bars with Exponential Variation of Area

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Abstract--- **A general equation for the evaluation of natural frequencies of axial vibrations of a bar which has two distinct portions i) with constant area of cross-section and ii) with exponential variation. This type of profile is used in machine components which include heat transfer applications such as fins and horns used for ultrasonic machining. The pillars of flyover bridges are also another potential application in civil engineering. The effect of the ratio of length parameter 'r' and exponential profile parameter 'β' on natural frequencies have been analyzed and presented graphically.**

Keywords--- Natural Frequency, Eigen Values, Mode, Variable Cross Section, Parameter

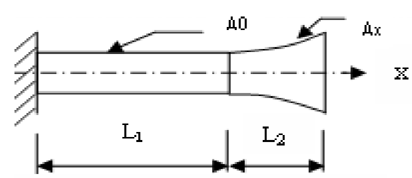
I. INTRODUCTION

Applications of bars with varying cross-section are common in many industrial applications. Common profiles used are cylindrical, conical, concave parabolic and convex parabolic. The fins which are extended surfaces to enhance the rate of heat transfer from the wall to an ambient medium through convective mode of heat transfer are effectively used in industrial equipment. These fins are of different profiles. The fins are subjected to wall excitation due to fluctuations of flow of fluid within the equipment may cause axial vibrations. Hence it is necessary to know the natural frequencies of spine fin configuration in order to avoid the undesirable effects of resonance. Results obtained by Lubkin and Luke [1] are applicable to rods having parabolic profile. Md. Amir Ali et.al. [2] have studied axial vibration analysis with conical profile. The efficiency of ultrasonic machining [USM] varies with the displacement amplitude of working face of the tool. The amplitude of vibration is increased by means of focusing element called "concentrator" or "horn". One end of the horn is fixed to the vibration generator (usually the bigger end) and tool is attached to the smaller end. The analysis of the USM horns with different profiles in respect of longitudinal vibrations are helpful in USM machining studies [3], [4].

Transverse vibrations of beams with linearly varying section was presented by Ward [5]. Exact solution for conical beams was studied by Conway [6] and obtained solution in terms of Bessel Functions. Bapat [7] obtained solutions for free longitudinal vibration of exponential and catenoidal bars. Exact solution for longitudinal natural frequencies of a variable cross-section rod with polynomial variation in cross sectional area and mass distribution along the member using exact element method was given by Eisenberger [8]. Kumar and Sujith [9] reported exact solutions for longitudinal vibration rods of non uniform cross section. Li [10] has presented exact solutions for longitudinal vibration of multistepped bars with cross section in terms of area.

In present work, longitudinal vibration of a fixed - free slender rod of variable cross-section is investigated. The model chosen is a slender rod of partly uniform and partly of variable cross-section. The effect of the ratio of length of the uniform portion to the length of the exponentially varying area 'r'on the natural frequency has been theoretically analyzed. The exponential profile parameter 'β' is varied and significance of this parameter on natural frequencies has been studied to keep the operating conditions of the equipment away from the resonance condition. Results presented in this investigation can be usefully applied for transducers where the exponential area variation in combination with a uniform portion exists.

II. ANALYSIS



Configuration Diagram

The differential equation expressed in terms of displacement function u(x, t) is $$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{1}{A(x)}\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] \quad (1)$$

Let $u = v(x)e^{\omega_n t}$ (2)

Equation (1) can be expressed as

$$\frac{d^2 v(x)}{dx^2} + \frac{1}{A(x)}\frac{dA(x)}{dx}\frac{dv(x)}{dx} + \omega_n^2\left(\frac{\rho}{E}\right)v(x) = 0 \quad (3)$$

Bar shown in configuration diagram consists of a uniform portion of length L_1 and varying cross sectional area with exponential variation of length L_2

(i) Uniform cross sectional area A_0 for bar of length L_1: $0 \le x \le L_1$

The equation of motion Eq. (3) reduces to

$$v'' + \omega_n^2 (\frac{\rho}{E}) v = 0 \tag{4}$$

Where primes denote differentiation with respect to x

Defining $\eta = \omega_n \sqrt{\frac{\rho}{E}}$

Eq.(4) now becomes $v'' + \eta^2 v = 0$ (5)

The solution to the above equation is

$$v = B_1 \operatorname{Sin} \eta x + B_2 \operatorname{Cos} \eta x \tag{6}$$

Where B_1 and B_2 are constants of integration

(ii) Exponential variation of area for bar of length L_2: $L_1 \le x \le (L_1+L_2)$

$$A(x) = A_0 e^{-\beta \frac{(x-L_1)}{L_2}} \tag{7}$$

β is a parameter to be determined depending on the profile

$$\frac{dA(x)}{dx} = \frac{-\beta A_0}{L_2} e^{-\beta \frac{(x-L_1)}{L_2}} \tag{8}$$

$$\frac{dA(x)}{dx} = \frac{-\beta}{L_2} A(x) \tag{9}$$

The equation of motion for the portion of the bar from Eq. (3) is

$$v'' - \frac{\beta}{L_2} v' + \eta^2 v = 0 \tag{10}$$

Defining $v = Ce^{+d\frac{(x-L_1)}{L_2}}$ (11)

The Eq.(10) transforms to

$$\frac{d^2}{(L_2)^2} - (\frac{\beta}{L_2})(\frac{d}{L_2}) + \eta^2 = 0 \tag{12}$$

$$d^2 - \beta d + \eta^2 L_2^2 = 0 \tag{13}$$

where d can be obtained in terms of β, η, L_2

$$d_1 = \frac{\beta}{2} + i \frac{L_2}{2} \sqrt{4\eta^2 - \frac{\beta^2}{L_2^2}} \tag{14}$$

$$d_2 = \frac{\beta}{2} - i \frac{L_2}{2} \sqrt{4\eta^2 - \frac{\beta^2}{L_2^2}} \tag{15}$$

The solution for varying cross section part of the bar is

$$v = e^{\frac{\beta(x-L_1)}{2L_2}} \left(C_1 \operatorname{Cos} \frac{c(x-L_1)}{L_2} + C_2 \operatorname{Sin} \frac{c(x-L_1)}{L_2} \right) \tag{16}$$

Where $c^2 = \frac{L_2^2}{4} \left[4\eta^2 - \frac{\beta^2}{L_2^2} \right]$ (17)

The solutions for the bar are expressed as

(i) Uniform portion: $0 \le x \le L_1$

$v = B_1 \sin \eta x + B_2 \cos \eta x$ (18)

(ii) Exponentially varying area: $L_1 \le x \le (L_1 + L_2)$

$$v = e^{\frac{\beta(x-L_1)}{2L_2}}\left(C_1 \mathrm{Cos}\,\frac{c(x-L_1)}{L_2} + C_2 \mathrm{Sin}\,\frac{c(x-L_1)}{L_2}\right) \tag{19}$$

$$\frac{dv}{dx} = e^{\frac{\beta(x-L_1)}{2L_2}}\left\{\frac{\beta}{2L_2}\left(C_1 \mathrm{Cos}\frac{c(x-L_1)}{L_2} + C_2 \mathrm{Sin}\,\frac{c(x-L_1)}{L_2}\right) + \frac{c}{L_2}\left(-C_1 \mathrm{Sin}\,\frac{c(x-L_1)}{L_2} + C_2 \mathrm{Cos}\frac{c(x-L_1)}{L_2}\right)\right\} \tag{20}$$

The Constants B1, B2, C1 and C2 are determined as given below.

a) At x = 0, v = 0, From Eq.(18), B2 = 0

v = B1Sin(ηx) (21)

$$\frac{dv}{dx} = \eta B_1 \mathrm{Cos}(\eta x) \tag{22}$$

b) i) At x = L1 , v should be equal at the interface between both portions of the bar. From Eqs.(19) and (21)

B1Sin(η L1) = C1 (23)

ii) At x = L1 , strain $\frac{dv}{dx}$ should also be equal at the interface. From Eqs. (20) and (22)

$$\eta B_1 \mathrm{Cos}(\eta L_1) = C_1\left(\frac{\beta}{2L_2}\right) + C_2\left(\frac{c}{L_2}\right) \tag{24}$$

Substituting for B1 from Eq. (23)

$$C_1\left[\eta \frac{\mathrm{Cos}(\eta L_1)}{\mathrm{Sin}(\eta L_1)}\right] = C_1\left(\frac{\beta}{2L_2}\right) + C_2\left(\frac{c}{L_2}\right) \tag{25}$$

$$C_1\left[\eta \mathrm{Cos}(\eta L_1) - \frac{\beta}{2L_2}\mathrm{Sin}(\eta L_1)\right] = C_2\left(\frac{c}{L_2}\right)\mathrm{Sin}(\eta L_1) \tag{26}$$

c) At x = (L1+L2) strain $\frac{dv}{dx} = 0$. From Eq. (20)

$$\frac{\beta}{2L_2}\left[C_1 \mathrm{Cos}(c) + C_2 \mathrm{Sin}(c)\right] + \frac{c}{L_2}\left[-C_1 \mathrm{Sin}(c) + C_2 \mathrm{Cos}(c)\right] = 0 \tag{27}$$

$$C_1\left[\frac{\beta}{2L_2}\mathrm{Cos}(c) - \frac{c}{L_2}\mathrm{Sin}(c)\right] + C_2\left[\frac{\beta}{2L_2}\mathrm{Sin}(c) + \frac{c}{L_2}\mathrm{Cos}(c)\right] = 0 \tag{28}$$

Substituting for C1 from Eq. (26)

$$\frac{c}{L_2}\mathrm{Sin}(\eta L_1)\left[\frac{\beta}{2L_2}\mathrm{Cos}(c) - \frac{c}{L_2}\mathrm{Sin}(c)\right] + \left[\eta \mathrm{Cos}(\eta L_1) - \frac{\beta}{2L_2}\mathrm{Sin}(\eta L_1)\right]\left[\frac{\beta}{2L_2}\mathrm{Sin}(c) + \frac{c}{L_2}\mathrm{Cos}(c)\right] = 0 \tag{29}$$

Simplifying the above Eq. (29) we get

$$\eta L_2 \mathrm{Cos}(\eta L_1)\left[\frac{\beta}{2}\mathrm{Sin}(c) + c\mathrm{Cos}(c)\right] - \left(c^2 + \frac{\beta^2}{4}\right)\mathrm{Sin}(\eta L_1)\mathrm{Sin}(c) = 0 \tag{30}$$

$$\because c^2 = \frac{L_2^2}{4}\left[4\eta^2 - \frac{\beta^2}{L_2^2}\right]$$

$$c^2 = \eta^2 L_2^2 - \frac{\beta^2}{4}$$

$$\therefore \left(c^2 + \frac{\beta^2}{4}\right) = \eta^2 L_2^2 \tag{31}$$

$$\left(4c^2+\beta^2\right)=4\eta^2L_2^2$$

$$\sqrt{4c^2+\beta^2}=2\eta L_2 \tag{32}$$

$$\left[\beta \mathrm{Tan}(c)+2c\right]-2\eta L_2 \mathrm{Tan}(\eta L_1)\mathrm{Tan}(c)=0 \tag{33}$$

$$\left[\beta \mathrm{Tan}(c)+2c\right]-\sqrt{4c^2+\beta^2}\,\mathrm{Tan}\left[\frac{\sqrt{4c^2+\beta^2}}{2}\left(\frac{L_1}{L_2}\right)\right]\mathrm{Tan}(c)=0 \tag{34}$$

Defining $\delta=\sqrt{\left(4c^2+\beta^2\right)}$ and $r=\left(\frac{L_1}{L_2}\right)$

The frequency equation is

$$\left[\beta \mathrm{Tan}(c)+2c\right]-\delta\left[\mathrm{Tan}\left(\frac{\delta.r}{2}\right)\right]\mathrm{Tan}(c)=0 \tag{35}$$

Let $P = \beta\ \mathrm{Tan}\ (c) + 2c$

$Q = \delta\left[\mathrm{Tan}\left(\frac{\delta.r}{2}\right)\right]\mathrm{Tan}(c)$

Frequency equation can be written in a simplified form as,

$$P - Q = 0 \tag{36}$$

The roots of the equation will give eigenvalues (c).

If $L_1 = 0$, then $r = 0$ and the frequency equation (35) reduces to

$$P = 0 \quad \text{i.e., } \mathrm{Tan}(c) = -\frac{2c}{\beta}. \tag{37}$$

which is verified with the available literature [10].

III. Results & Discussions

The roots of the frequency equation (35) yield the eigenvalues for the longitudinal vibration of the bar with varying cross section. There are two parameters which are explicit from the frequency equation i) ratio of length of uniform portion L_1 to the length of exponentially varying portion L_2 defined as 'r' ii) profile parameter 'β' expressing the variation of area.

The roots of equation (35) have been obtained by using EXCEL program for first five modes, varying 'β' and 'r'. Results thus obtained have been presented graphically

Table 1 Eigen values for mode1					
β↓	r=0.0	r=0.25	r=0.50	r=0.75	r=1.0
1	1.8366	1.4762	1.1917	2.6981	2.3899
2	2.0288	1.5964	1.1716	2.6815	2.3938
3	2.1746	1.6423	0.9489	2.6541	2.3755
4	2.2889	1.6164	2.9765	2.6233	2.3309

Table 2 Eigen values for mode2					
β↓	r=0.0	r=0.25	r=0.50	r=0.75	r=1.0
1	4.8158	3.7969	5.2764	4.535	3.9398
2	4.9132	3.8018	5.3037	4.5537	3.911
3	5.0036	3.7815	5.3195	4.5415	3.8361
4	5.087	3.7345	5.3251	4.4938	3.7117

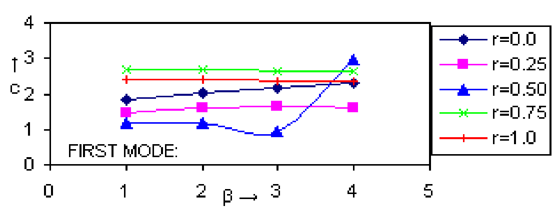


Fig. 1 Variation of Eigen values

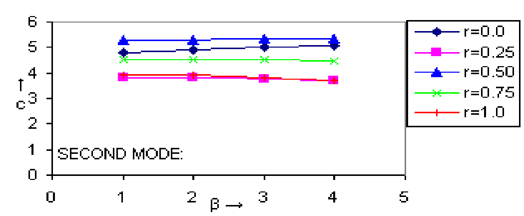


Fig. 2 Variation of Eigen values

Table 3 Eigen values for mode3

β↓	r=0.0	r=0.25	r=0.50	r=0.75	r=1.0
1	7.9171	8.8098	7.3581	6.2747	5.5106
2	7.9787	8.8187	7.3721	6.2495	5.5057
3	8.0385	8.8238	7.3711	6.2087	5.4859
4	8.0962	8.8258	7.3535	6.1545	5.4531

Table 4 Eigen values for mode4

β↓	r=0.0	r=0.25	r=0.5	r=0.75	r=1.0
1	11.0408	11.3394	11.5374	8.1055	7.0765
2	11.0855	11.3648	11.5492	8.12	7.0636
3	11.1295	11.3859	11.5549	8.1223	7.0283
4	11.1727	11.4031	11.5549	8.1121	6.969

Table 5 Eigen values for mode5

β↓	r=0.0	r=0.25	r=0.5	r=0.75	r=1.0
1	14.1724	13.8473	13.6287	9.8732	8.6472
2	14.2074	13.8677	13.637	9.8608	8.6427
3	14.2421	13.884	13.6382	9.8357	8.6269
4	14.2764	13.8961	13.6318	9.7977	8.6011

Table 6 Eigen values for r = 0

β↓	Mode1	Mode2	Mode3	Mode4	Mode5
1	1.8366	4.8158	7.9171	11.0408	14.1724
2	2.0288	4.9132	7.9787	11.0855	14.2074
3	2.1746	5.0036	8.0385	11.1295	14.2421
4	2.2889	5.087	8.0962	11.1727	14.2764

Table 7 Eigen values for r = 0.25

β↓	Mode1	Mode2	Mode3	Mode4	Mode 5
1	1.4762	3.7969	8.8098	11.3394	13.8473
2	1.5964	3.8018	8.8187	11.3648	13.8677
3	1.6423	3.7815	8.8238	11.3859	13.884
4	1.6164	3.7345	8.8258	11.4031	13.8961

Table 8 Eigen values for r = 0.50

β↓	Mode1	Mode2	Mode3	Mode4	Mode5
1	1.1917	5.2764	7.3581	11.5374	13.6287
2	1.1716	5.3037	7.3721	11.5492	13.637
3	0.9489	5.3195	7.3711	11.5549	13.6382
4	2.9765	5.3251	7.3535	11.5549	13.6318

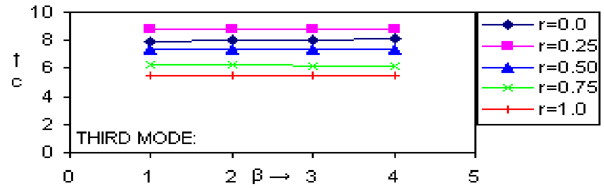


Fig. 3 Variation of Eigen values

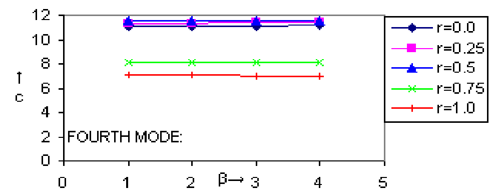


Fig. 4 Variation of Eigen values

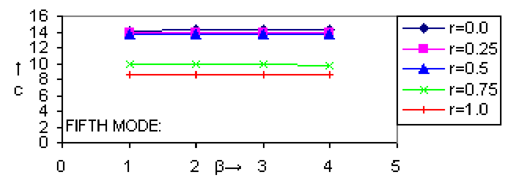


Fig. 5 Variation of Eigen values

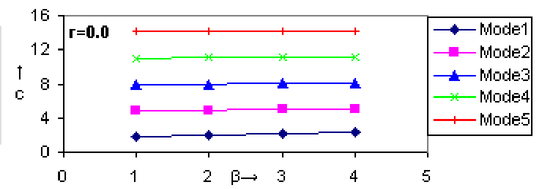


Fig. 6 Variation of Eigen values

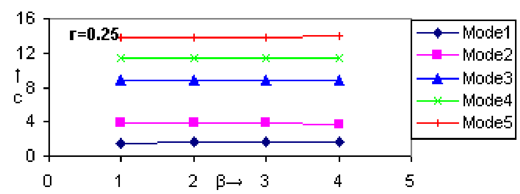


Fig. 7 Variation of Eigen values

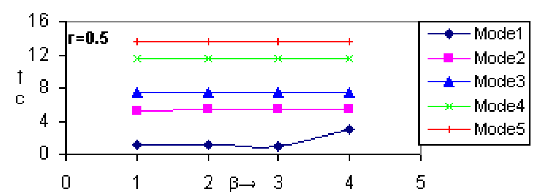


Fig. 8 Variation of Eigen values

Table 9 Eigen values for r = 0.75

β↓	Mode1	Mode2	Mode3	Mode4	Mode5
1	2.6981	4.535	6.2747	8.1055	9.8732
2	2.6815	4.5537	6.2495	8.12	9.8608
3	2.6541	4.5415	6.2087	8.1223	9.8357
4	2.6233	4.4938	6.1545	8.1121	9.7977

Table 10 Eigen values for r = 1.0

β↓	Mode1	Mode2	Mode3	Mode4	Mode5
1	2.3899	3.9398	5.5106	7.0765	8.6472
2	2.3938	3.911	5.5057	7.0636	8.6427
3	2.3755	3.8361	5.4859	7.0283	8.6269
4	2.3309	3.7117	5.4531	6.969	8.6011

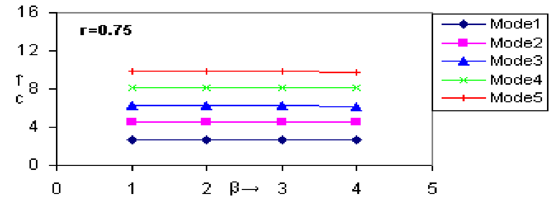


Fig. 9 Variation of Eigen values

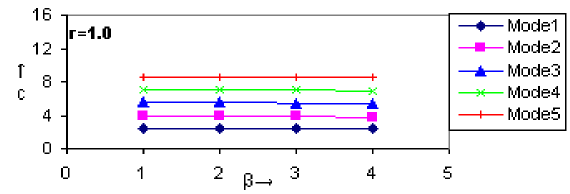


Fig. 10 Variation of Eigen values

from figure 1 to 10 and tabulated from table 1 to 10 by calculating the value of ‘c’ up to fourth decimal place shown in tables.

Figures 1 to 5 show the variation of ‘c’ with increasing value of ‘β’ for various values of ‘r’. Distinct effect of parameter r = 0.5 can be seen in first mode and has the highest eigenvalue when β = 4. Eigenvalues are closely clustered for r = 0, 0.25 and 0.5 for fourth and fifth modes. For third mode the values of ‘c’ are approximately same for all ‘β’ values at a parameter value of ‘r’.

The variation of eigenvalues (c) is graphically shown in figures 6 to 10 for different modes at a particular value of ‘r’. All figures indicate similar variation of ‘c’ for all values of ‘r’ at different modes, except for the first mode when r = 0.5.

IV. CONCLUSIONS

Parameters ‘β’ and ‘r’ have significant effect on the eigenvalues of axial vibrations of varying cross section bar. The effect of increasing ‘r’ decreases the eigenvalues for different modes (Figs.6 to10). When r = 0.5 the variation of eigenvalue is steep for the value of ‘β’ between 3 and 4 (Fig.8).

From fourth mode to fifth mode, it is observed that the eigenvalues are very close for r = 0.0, 0.25 and 0.5 as evident from figures 4 and 5.

Changing the ratio of lengths ’r’ and profile parameter ‘β’ will contribute to the resulting changes in stiffness and mass of the bar. These changes in stiffness and mass do affect the eigenvalues.

REFERENCES

[1] Lubkin.J.L.,Luke.Y.L.,1953, “Frequencies of longitudinal vibration of a slender rod of a variable section”, Journal of Applied Mechanics, 20, No.2, pp. 173-176.

[2] Md.Amir Ali, Subba Rao.V., Sarean.M.M.M.,1979, “Longitudinal vibrations in a fixed free slender rod with uniform taper”, Indian Journal of Technology, 17, pp.335-338.

[3] Seah.K.H.W., Wang.Y.S., Lee.L.C., 1992, “Design of tool holders for ultrasonic machining using FEM”, National University of Singapore.

[4] Murti.V.S.R., Rama Murty.G., Nagaprasada Rao.B., Surendar.Ch., 2002, “Design and Analysis of USM Horns of various profiles using FEM”, Proceedings of National Conference on Recent trends in CAD/CAM, Dept. of Mech.Engg., JNTU college of Engg., Kakinada (India), pp. 188-192.

[5] Ward.P.F., 1913, “Transverse vibration of bars of varying cross sections”, Philos. Mag., 25, pp. 86-106.

[6] Conway.H.D., Beeker.E.C.H. and Dubil.F.H., 1964, “Vibration frequencies of tapered bars and circular plates”, ASME:J.Appl.Mech., 31, pp. 329-331.

[7] Bapat.C.N., 1995, “Vibration of rods with uniformly tapered sections”, J.Sound.Vib., 185, pp. 185-189.

[8] Eisenberger.M., 1991, “Exact longitudinal vibration Frequencies of a variable cross section Rod”, Appl.Acoust., 34, pp.123-130.

[9] Kumar.B.M., and Sujith.R.L., 1997, “Exact solutions for the longitudinal vibrations of non uniform Rods”, J.Sound. Vib., 207, No.5, pp. 721-729.

[10] Li.Q.S., 2000, “Exact solution for longitudinal vibration of Multi step Bars with varying cross section”, J.Sound. Vib., 122, pp. 183-187.