The Numerical Solution of Differential Transforms Method for Some Non-Linear System of Equations

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Abstract: In this paper, we present the definition and operation of the differential transform and investigate with particular example in exact solutions, Lorenz systemz and chua's system of nonlinear differential equations .The numerical results of the present method are presented.

I. INTRODUCTION

The differential transform method is one of the approximate methods which can be easily applied to many linear and nonlinear problems and is capable of reducing the size of computational work. Exact solutions can also be achieved by the known forms of the series solutions. The concept of the differential transformation method has been introduced to solve linear and nonlinear initial value problems in electric circuit analysis The differential transformation method is a semi numerical analytic technique that formalizes the Taylor series in a totally different manner. With this method, the given differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. There is no need for linearization or perturbations, large computational work and round-off errors are avoided.

The differential transform method has solution in the form of polynomials. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken a long time for large orders. The present method reduces the size of computational domain and applicable to many problems easily. When dealing with non-linear systems of ordinary differential equations, it is often the case that a closed form analytic solution for the system of interest is normally unobtainable.

II. DIFFERENTIAL TRANSFORM

Definition 1. :

Consider the analytical function of one variable $u(x)$ which is defined on $D = [0, X] \subseteq \mathbb{R} \; and \; x_0 \in D$ One-dimensional differential transform of $u(x)$ denoted by $U(K)$ and is defined on $\mathbb{N} \cup \{0\}$ as the following:

Definition 1: The one-dimensional differential transform of function $u(x)$ is defined as follows:

$$U(K) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0} \qquad (2.1)$$

Where $u(x)$ is the original function and is $U(K)$ called the transformed function

Definition 2:
Inverse differential transform of $U(K)$ in the Eq. (1) is defined as follows:

$$u_0(x) = \sum_{k=0}^{\infty} U(K)(x - x_0)^k \qquad (2.2)$$

Since $u(x)$ is an analytical function, it is clear that $u(x) = u_0(x)$ By combination of Eqs. (1) and (2), with $x_0 = 0$ the function $u(x)$ can be written as

Definition 3:

$$u_0(x) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0} \qquad (2.3)$$

Definition 4:

The differential inverse transform of $U(K)$ is defined as follows:

$$u_0(x) = \sum_{k=0}^{\infty} U(K)(x)^k \qquad (2.4)$$

Substituting (2.3) into (2.4) we have

$$u_0(x) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0} x^k \qquad (2.5)$$

In real applications, the function $u(x)$ by a finite series of (4) can be written as

$$u(x) = \sum_{k=0}^{\infty} U(K)(x)^k$$

and (2.4) implies that

$$u(x) = \sum_{k=n+1}^{\infty} U(K)(x)^k \quad (2.6)$$

is neglected as it is small. Usually, the values of n are decided by a convergency of the series coefficients. From the definition of (3) and (4), it is readily proved that the transformed functions comply with the basic mathematical operations.
The fundamental mathematical properties of one-dimensional differential transform can readily be obtained and are summarized in the following theorem.

Theorem2.1:
If $U(K), F(K) anf\ G(K)$ are one-dimensional differential transforms of the functions $u(x), f(x), g(x)$ respectively then,

1. $If\ u(x) = f(x) \mp g(x)$ then $U(K) = F(K) \mp G(K)$
2. $If\ u(x) = \alpha f(x)$ then $U(K) = \alpha F(K)$
3. $If\ u(x) = f(x)g(x)$ then $U(K) = \sum_{l=1}^{k} F(l)G(k-l)$
4. $If\ u(x) = \frac{df(x)}{dx}$ then $U(K) = (k+1)F(k+1)$
5. $If\ u(x) = \frac{d^m f(x)}{dx^m}$ then $U(K) = (k+1)(k+2\ldots\ldots(k+m)F(k+m)$
6. $If\ u(x) = \int_0^x f(t)dt$ then $U(K) = \frac{F(K-1)}{K}, K \geq 1, U(0) = 0.$
7. $If\ u(x) = x^m$ then $U(K) = \delta(k-m)\begin{cases}1 & k=m\\ 0 & 0.W\end{cases}$
8. $If\ u(x) = \sin(\omega x + \alpha)$ then $U(K) = \frac{\omega^k}{k!}\sin(\frac{k\pi}{2} + \alpha)$
9. $If\ u(x) = \cos(\omega x + \alpha)$ then $U(K) = \frac{\omega^k}{k!}\cos(\frac{k\pi}{2} + \alpha)$

III. CONSIDER THE FOLLOWING NON-LINEAR DIFFERENTIAL SYSTEM:

$$\frac{dx(t)}{dt} + \frac{dy(t)}{dt} + x(t) + y(t) = 1 \quad (3.1)$$

$$\frac{dy(t)}{dt} = 2x(t) + y(t) \quad (3.2)$$

With initial condition $x(0) = 0, y(0) = 1$

Taking the differential transform method to Eqs. (3.1) and (3.2), we obtain

$$(K+1)X(K+1) + (K+1)Y(K+1) + X(K) + Y(K) = 1 \quad (3.3)$$

$$(K+1)Y(K+1) = 2X(K) + Y(K) \quad (3.4)$$

Rearranging Eq. (3.3) and (3.4) can be written as:

$$X(K+1) = \frac{1}{(K+1)}[(K+1)Y(K+1) - X(K) - Y(K) - 1] \quad (3.5)$$

$$Y(K+1) = \frac{1}{(K+1)}[2X(K) + Y(K)] \quad (3.6)$$

with initial conditions, the numerical results for the differential transformation method are presented in Table 1.Differential transformation values of our example, for k = 0,1,2,3…

k	K+1	X(k+1)	Y(k+1)
0	1	-1	1
1	2	1	-1/2
2	3	-5/6	½
3	4	5/8	-7/24
4	5	-7/120	23/120
5	6	-19/144	1/180
.	.	.	.
.	.	.	.
.	.	.	.

Table 1

$$x(t) = \sum_{k=0}^{6} X(k)t^k \quad (3.7)$$

$$y(t) = \sum_{k=0}^{6} Y(k)t^k \quad (3.8)$$

Substituting all values of X(K)and Y(K) into Eqs. (3.7) and (3.8), we obtain

$$x(t) = -t + t^2 - \frac{5}{6}t^3 + \frac{5}{8}t^4 - \frac{7}{120}t^4 - \frac{19}{144}t^6 + \cdots \quad (3.9)$$

$$y(t) = 1 + t - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{7}{24}t^4 + \frac{23}{120}t^4 + \frac{1}{180}t^6 + \cdots \quad (3.10)$$

IV. LORENZ SYSTEM

Here the system we are interest in is the Lorenz system. As is well-known, the Lorenz system does not admit a closed form solution and moreover it can exhibit both chaotic and non-chaotic behavior for distinct parameter values:

$$\frac{dx}{dt} = \sigma(y - x) \quad (4.1)$$

$$\frac{dy}{dt} = Rx - xz - y \quad (4.2)$$

$$\frac{dz}{dt} = xy + bz \quad (4.3)$$

where *x*, *y* and *z* are, respectively, proportional the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow; while *σ*, *b* and the so-called bifurcation parameter *R* are real constants. Throughout this study, we set *σ* = 10, *b*

$= -8/3$ and vary the parameter R. It is well known that chaos sets in around the critical parameter value $R = 24.74$.

Taking the differential transform of Esq. (1)– (3) with respect to time t gives

$$X(K+1) = \frac{H}{K+1}\left[-\sigma X(k) + \sigma Y(k)\right] \quad (4.4)$$

$$Y(K+1) = \frac{H}{K+1}\left[RX(k) - \sum_{l=0}^{k} X(l)Z(k-l) - Y(k)\right] \quad (4.5)$$

$$Z(K+1) = \frac{H}{K+1}\left[\sum_{l=0}^{K} X(l)Y(k-l) + BZ(k)\right] \quad (4.6)$$

where $X(k)$, $Y(k)$ and $Z(k)$ are the differential transformations of the corresponding functions $x(t)$, $y(t)$ and $z(t)$, respectively, and the initial conditions are given by $X(0) = -15.8$, $Y(0) = -17.48$ and $Z(0) = 35.64$.

The difference equations presented in Eqs. (4.4)–(4.6) describe the Lorenz system, from a process of inverse differential transformation, i.e

$$x_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k X_i(k), 0 \le t \le H_i \quad (4.7)$$

$$y_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k Y_i(k), 0 \le t \le H_i \quad (4.8)$$

$$z_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k Z_i(k), 0 \le t \le H_i \quad (4.9)$$

Where $k = 0, 1, 2, \ldots, n$ represents the number of terms of the power series, $i = 0, 1, 2, \ldots$, expresses the i^{th} sub-domain and H_i is the sub-domain interval.

$$x(t) = -\frac{79}{5} - \frac{84}{5}t - \frac{5625}{50}t^2 + \frac{744769}{750}t^3 - \frac{1601549659}{4500}t^4 + \cdots$$

$$y(t) = -\frac{437}{25} - \frac{52323}{250}t - \frac{3570919}{2500}t^2 + \frac{1144338422}{86411}t^3 - \frac{2074616165}{18447}t^4 + \cdots$$

$$z(t) = \frac{891}{25} - \frac{22643}{125}t - \frac{13110751}{1700}t^2 + \frac{781669993}{56491}t^3 - \frac{3690250604}{33465}t^4 + \cdots$$

V. CHUA'S SYSTEM

Chaos is a Mathematical term that describes complex behavior in deterministic dynamical systems, which has short-term predictability but is nevertheless unstable and unpredictable in the long term. Extensive studies on chaos in the 1980s clarified that chaos is ubiquitous not only in Mathematical models but also in real-world systems. Any dynamical systems can be modeled by any of the following systems described. A linear system is a mathematical model of a system based on the use of linear operator. A non-linear system is a system which is not linear, that is a system which does not satisfy the superposition principle or whose output is not directly proportional to its input. The third effective system is a Hybrid system, which is a dynamic system that exhibits both continuous and discrete dynamic behavior-(i.e) a system that can be described by a differential equation .

An application of hybrid dynamical systems to biology and medicine, particularly, cancer and its treatment are considered in focus on cancer dynamics of a colorectal crypt in colorectal cancer. Since cancer originates from a mutation initially creating a single abnormal cell, there exists a problem of small numbers of cells, which is different from that of small copy numbers of proteins and mRNAs discussed in Owing to the small numbers of cells, they study cell-based 'hybrid' models in which cells are considered as discrete, with dynamical changes such as division, proliferation and movement encoded either through rules or as the result of dynamics occurring on other scales. They also compare and contrast two different cell-based models with a homogenized continuum model.

When the number of cancer cells increases with the tumour growth, another hybrid model can be used by describing cells as discrete in some parts and as a continuum in others, Hybrid dynamical systems are usually described with continuous and discrete-state variables. Representations of time and space, however, are also quite important; the above modelling brings hybrid properties of space into sharp relief. The hybrid properties of time have also been discussed in previous studies .

Thus, the mathematical modelling of hybrid dynamical systems should also be considered from the perspective of time and space; that is, the continuous and discrete structure of time and space. A famous example of hybrid dynamical system which can be described as follows:

$$\frac{dx}{dt} = \alpha\left(y - x - h(x)\right) \quad (5.1)$$

$$\frac{dy}{dt} = x - y + z \quad (5.2)$$

$$\frac{dz}{dt} = -\beta y \quad (5.3)$$

where x , y, and z are continuous state variables, 'α' and 'β' are positive parameters and $h(x)$ is a piecewise smooth function of x . The cover image of this Theme Issue is composed of some strange attractors obtained from equations (5.1)–(5.3) with $h(x)$ that is more complicated than the original. Consider $\alpha = 10, \beta = 16.82$, $h(x) = -0.55x(|x+1| - |x-1|)$. Solve the system of equations .1), (2.2) & (2.3) by Differential Transform Method,

$X(K+1) = \frac{1}{k+1}\sigma[Y(K) - 0.02X(K)]$ (5.4)

$Y(K+1) = \frac{1}{k+1}X(K) - Y(K) + Z(K)$ (5.5)

$Z(K+1) = -\frac{1}{k+1}\beta Y(K)$ (5.6)

The power series obtained is given bellow

$x(t) = \sum_{k=0}^{10} X(k)t^k; y(t) = \sum_{k=0}^{10} Y(k)t^k; z(t) =$ $k=010Z(k)tk$ (5.4),(5.5)&(5.6)

For the initial values $x(0) = 0.01, y(0) = 0.01$ & $z(0) = 0.01$, in equations (5.1),(5.2)& (5.3)by using differential transform method, the values obtained for x , y and z are given in the following table

Table .5.1-Iteration values of x, y, z

Iteration	**x**	**y**	**z**
1	**0.01**	**0.01**	**0.01**
2	**0.049**	**0.005**	**-0.0841**
3	**0.0201**	**-0.0201**	**-0.0421**
4	**-0.1025**	**-0.0010**	**0.1690**
5	**0.0053**	**0.0338**	**0.0084**
6	**0.1685**	**-0.0101**	**-0.2843**
7	**-0.0674**	**-0.0529**	**0.0850**
8	**-0.2578**	**0.0353**	**0.4449**
9	**0.2023**	**0.0759**	**-0.2969**
10	**0.3593**	**-0.0853**	**-0.63783**

By taking values for t∈(0.0001,0.5) and using equations(5.4),(5.5)&(5.6), the sample values obtained out of nearly 5000 values are tabulated (following Table)for reference.In the power series analysis difference in the values of ‘x, y and z’ are of the system variable is calculated as reference and are plotted as shown below

s.no	**X**	**Y**	**Z**	$x_{i+1}-x_i$	$y_{i+1}-y_i$	$z_{i+1}-z_i$
1	0.01	0.01	0.00999159	0.0000049006	0.0000004994	-0.0000084113
2	0.0105	0.01005	0.00914647	0.0000049376	0.0000004592	-0.0000084902
3	0.01099	0.01009	0.008293954	0.0000049685	0.0000004190	-0.0000085591
4	0.01149	0.01013	0.007435056	0.0000049934	0.0000003789	-0.0000086178
5	0.01199	0.01017	0.006570783	0.0000050122	0.0000003390	-0.0000086665
6	0.01249	0.0102	0.005702135	0.0000050251	0.0000002993	-0.0000087052
7	0.013	0.01023	0.004830102	0.0000050321	0.0000002600	-0.0000087341
8	0.0135	0.01025	0.003955658	0.0000050334	0.0000002210	-0.0000087534
9	0.014	0.01027	0.00307976	0.0000050291	0.0000001825	-0.0000087631
10	0.0145	0.01029	0.002203345	0.0000050193	0.0000001446	-0.0000087636
11	0.01501	0.0103	0.001327328	0.0000050042	0.0000001072	-0.0000087551
12	0.01551	0.01031	0.000452594	0.0000049840	0.0000000704	-0.0000087379
13	0.016	0.01032	-0.000419999	0.0000049588	0.0000000343	-0.0000087123
14	0.0165	0.01032	-0.001289624	0.0000049289	-0.0000000010	-0.0000086786
15	0.01699	0.01032	-0.002155493	0.0000048944	-0.0000000356	-0.0000086371
16	0.01748	0.01031	-0.003016853	0.0000048555	-0.0000000693	-0.0000085884
17	0.01796	0.0103	-0.003872993	0.0000048125	-0.0000001022	-0.0000085328
18	0.01844	0.01029	-0.004723249	0.0000047655	-0.0000001343	-0.0000084707
19	0.01891	0.01027	-0.005567003	0.0000047148	-0.0000001655	-0.0000084027
20	0.01938	0.01026	-0.006403687	0.0000046606	-0.0000001957	-0.0000083294

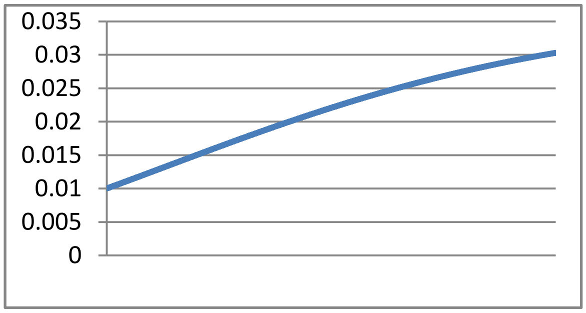

Fig.5. 1. X with $X_{i+1} - X_i$

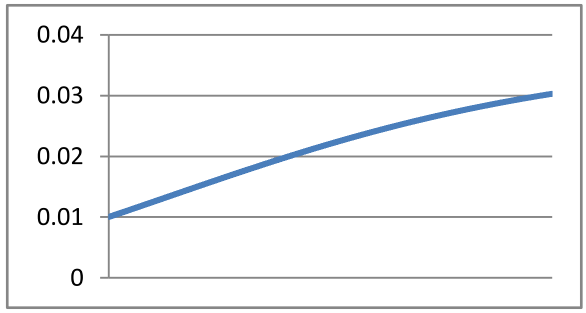

Fig. 5.2. X with $Y_{i+1} - Y_i$

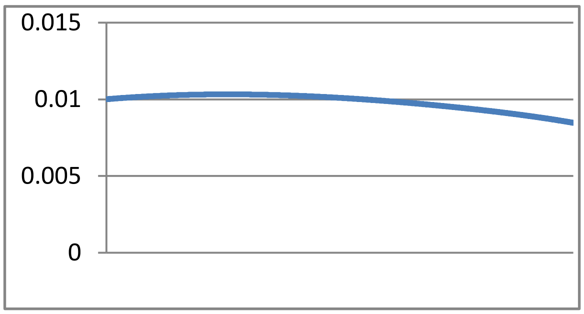

Fig. 5.3. Y with $X_{i+1} - X_i$

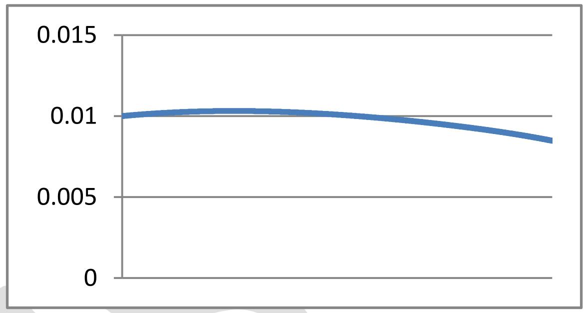

Fig. 5.4. Y with $Y_{i+1} - Y_i$

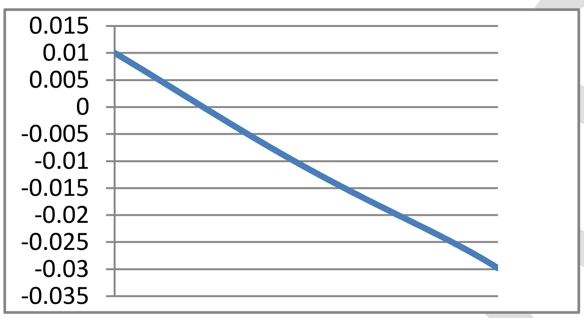

Fig. 5.5. Z with $X_{i+1} - X_i$

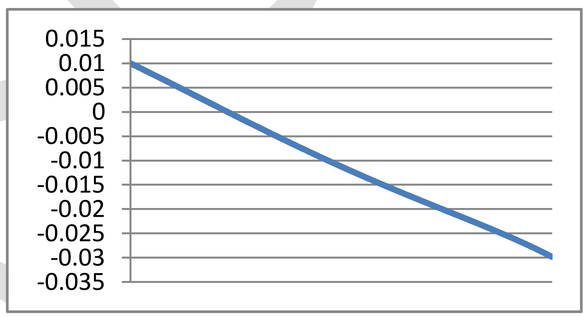

Fig.5. 6. Z with $Y_{i+1} - Y_i$

In Figures (5.1) and (5.2) ,the plot obtained for the system graph of variable '*x*' with x_{i+1}-x_i and y_{i+1}-y_i,show an exponential growth in the system. Figures (5.3) and (5.4), show the graph of the system variable '*y*' with x_{i+1}-x_i and y_{i+1}-y_i,Sow that the system growth is almost nil or towards the shrinking mode. Figures (5.5) and (5.6) the graph of the system variable '*z*' with x_{i+1}-x_i and y_{i+1}-y_i, show the system behavior along in the '*z*' direction.

VI. CONCLUSION

Apply the differential transform method in some non linear systems(particular example and Lorenz system) and also apply the DTM Method for the Dynamical Hybrid system(Chua's) is analyzed with the various possibilities of system variables x,y,z,, along there mutually perpendicular directions, we conclude that the graphs corresponding to the solution of Hybrid dynamical system growth is in only one direction and not in the other directions.

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