Approximation of Fourier Series of a Function Belonging to the Class \(Lip(\alpha)\) by \((\bar{N}, p_n, q_n)(E, s)\) - Summability Method

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Abstract: - Approximation of periodic functions by different linear summation methods have been studied by many researchers. Further, for sharpening the estimate of errors out of the approximations several product summability methods were introduced by different researchers. In this paper a new theorem has been established on \((\bar{N}, p_n, q_n)(E, s)\)-summability of Fourier series of a function belonging to \(Lip(\alpha)\)-class that generalizes several known results.

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I. INTRODUCTION

Approximation of periodic functions of a given period arose in connection with the convergence of Fourier series. The object of such study is to estimate the minimum error obtained out of the approximations of a function in a given interval. The most important trigonometric polynomials used in approximation theory are obtained by different linear summation methods of Fourier series of \(2\pi\) periodic function in the real line \(R\) (i.e., Cesàro mean, Nörlund mean, matrix mean etc.). Much advances in the theory of trigonometric approximation have been studied by different researchers for periodic function of Lipschitz class. The degree of approximation of functions belonging to \(Lip(\alpha), Lip(\alpha, r), Lip(\xi(t), r), (r \geq 1)\)-classes through trigonometric Fourier series expansion by using different summability method has been studied by various investigators like Mishra [3], Paikray [5], Lal [6] and Mishra et. al [8] and many others. Recently, Nigam [2] has established degree of approximation of a function belonging to weighted \(W(L_\alpha, \xi(t))\) class by \((C, 1)(E, q)\) mean. In an attempt to sharpening the estimate of errors and to have an advance study in this direction, here we have established a new theorem on \((\bar{N}, p_n, q_n)(E, s)\)-summability mean of Fourier series of a function \(f \in Lip(\alpha)\).

II. DEFINITIONS AND NOTATIONS

Let \(\sum u_n\) be a given infinite series with the sequence of partial sum \(\{s_n(f)\}\). Let \(\{p_n\}\) and \(\{q_n\}\) be sequences of positive real numbers such that,

\[ p_n = \sum_{k=0}^{n} p_k \quad \text{and} \quad q_n = \sum_{k=0}^{n} q_k \]

and let

\[ r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 \neq 0 \quad (p_1 = q_1 = r_1 = 0) \]

The sequence to sequence transformation (see [4]),

\[ t_n^N = \frac{1}{r_n} \sum_{k=0}^{n} p_n - q_k s_k \]

defines the \((N, p_n, q_n)\)-mean of the sequence \(\{s_n\}\) generated by the sequence of coefficients \(\{p_n\}\) and \(\{q_n\}\), and it is denoted by \(t_n^N\).

Similarly, we define the extended Riesz mean as the sequence to sequence transformation,

\[ t_n^N = \frac{1}{r_n} \sum_{k=0}^{n} p_k q_k s_k \quad \text{(2.1)} \]

where \( r_n = q_0 p_0 + q_1 p_1 + \ldots + q_n p_n \neq 0 \); \( p_1 = q_1 = r_1 = 0 \) defines the \((N, p_n, q_n)\)-mean of sequence \(\{s_n\}\) generated by the sequence of coefficients \(\{p_n\}\) and \(\{q_n\}\), and let it be denoted by \(t_n^N\).

If \( \lim_{n \to \infty} t_n^N \to s \) then the series \( \sum u_n \) is \((N, p_n, q_n)\) summable to \(s\).

Analogous to regularity conditions of Riesz summability [1], we have the necessary and sufficient conditions for regularity of \((\bar{N}, p_n, q_n)\) summability are

1. \( \frac{p_k}{r_n} \rightarrow 0 \), \( f \) or each integer \( k \geq 0 \) as \( n \to \infty \) and
2. \( \frac{1}{r_n} \sum_{k=0}^{n} p_k q_k < C |r_n| \)

where \( C \) is any positive integer independent of \(n\).

The sequence to sequence transformation,
\[ E^q_n = \frac{1}{(1+q)n} \sum_{t=0}^n a(t^n) q^{-n} s_n \quad (2.2) \]
defines the sequence \( \{E^q_n\} \) of \((E, q)\) mean of the sequence \( \{s_n\} \).

If \( E^q_n \to s \) as \( n \to \infty \) then \( \sum u_n \) summable to \( s \) with respect to \((E, q)\) summability and \((E, q)\) method is regular (see [1]).

Now we define, a new composite transformation \((\bar{N}, p_n, q_n)\) over \((E, s)\) of \( \{s_n\} \) as,

\[ T_{RE}^n = \frac{1}{r^n} \sum_{k=0}^n p_k q_k (E^s_k) = \frac{1}{r^n} \sum_{k=0}^n p_k q_k \left( \frac{1}{(1+q)^k} \sum_{v=0}^{k-1} a(v^n) \right) s^{k-v} s \quad (2.3) \]

If \( T_{RE}^n \to s \) as \( n \to \infty \), then \( \sum u_n \) summable to \( s \) by \((\bar{N}, p_n, q_n)\) \((E, s)\) summability method.

Let \( s_n \to s \) implies \( E^s_k(s_n) \to s \) as \( n \to \infty \). Hence \((E, s)\) method is regular. Now we may write, \( T_{RE}^n = t^n \left( E_n^s(s_n) \right) \to s \) as \( n \to \infty \). Therefore, \((\bar{N}, p_n, q_n)\) \((E, s)\) method is also regular.

**Remark 1.** If we put \( p_n = 1 \), in equation (2.1) then \((\bar{N}, p_n, q_n)\)-summability method reduces to \((\bar{N}, p_n)\)-summability and for \( p_n = 1 \), it reduces to \((\bar{N}, q_n)\)-summability. If we put \( p_n = 1 \) then \((\bar{N}, p_n)\)-summability reduces to \((C, 1)\)-summability.

Let \( f \) be \( 2\pi \) periodic and belonging to \( L^r [0, 2\pi] \), \( r \geq 1 \), be a signal function with the partial sum \( s_n(f) \), defined by

\[ s_n(f) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \quad (2.4) \]

A signal (function) \( f \in Lip(\omega) \), if

\[ \left| f(x) - f(x^t) \right| = O(t^\omega) \quad \text{for} \ 0 < \omega \leq 1, \ t > 0. \]

The \( L^r \)-norm of a function \( f : R \to R \) is defined by

\[ \|f\|_r = \text{sup} |f(x)| : x \in R \]

and \( L^r \)-norm of a function \( f : R \to R \) is defined by

\[ \|f\|_r = \left( \int_{[0,2\pi]} |f(x)|^r dx \right)^{\frac{1}{r}}, \ r \geq 1. \]

The degree of approximation of a function \( f : R \to R \) by a trigonometric polynomial \( (t_n) \) of order \( n \) under \( \|\cdot\|_\infty \) is defined by

\[ \|t_n - f(x)\|_\infty = \text{sup} |t_n(x) - f(x)| : x \in R \]

and the error \( E_n(f) \) of a function \( f \in L_r \) is given by

\[ E_n(f) = \text{min} \|t_n - f\|_r \]

We use the following notations throughout the paper:

\[ \phi(t) = f(x + t) + f(x - t) - 2f(x) \]

\[ K_n(t) = \frac{1}{2\pi n} \sum_{k=0}^{n} p_k q_k \left( \frac{1}{(1+q)^n} \sum_{v=0}^k a(v^n) \right) q^{-k-v} s_v \]

**III. KNOWN THEOREM**

**Theorem-1.** [2] If a function \( f \) is \( 2\pi \) periodic belonging to the class \( Lip(\alpha) \), then its degree of approximation by \((C, 1)(E, q)\)-summability mean of Fourier series is given by

\[ \|C^1 E^q_n - f\|_\infty = O \left( (n+1)^{-\alpha} \right), \ (0 < \alpha < 1). \quad (3.1) \]

Where \( C^1 E^q_n \) represents the \((C, 1)\) transform of \((E, q)\) transform of \( s_n(f) \).

**IV. MAIN THEOREM**

**Theorem-2** If a function \( f \) is \( 2\pi \) periodic belonging to the class \( Lip(\alpha) \), then its degree of approximation by \((\bar{N}, p_n, q_n)\)-summability mean of Fourier series \( s_n(f) \) is given by

\[ \|T_{RE}^n - f\|_\infty = O \left( (n+1)^{-\alpha} \right), \ (0 < \alpha < 1). \quad (4.1) \]

Where \( T_{RE}^n \) represents the \((\bar{N}, p_n, q_n)\) transform of \((E, s)\) transform of \( s_n(f) \).

To prove the above theorem, first we need to prove the following lemmas.

**Lemma-1.** \( |K_n(t)| = O(n), \ \text{for} \ 0 \leq t \leq \frac{1}{n+1}. \)

**Proof.** For \( 0 \leq t \leq \frac{1}{n+1} \), we have \( \sin nt \leq n \sin t \), thus

\[ |K_n(t)| = \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{(1+q)^n} \sum_{v=0}^{k} a(v^n) \right) q^{-k+v} \sin \left( \frac{v+1}{2} \right) \right| \]

\[ \leq \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{1+q} \right) \sum_{v=0}^{k} a(v^n) q^{-k+v} (2v+1) \sin \frac{v+1}{2} \sin \frac{v}{2} \right| \]

\[ \leq \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{1+q} \right) \sum_{v=0}^{k} a(v^n) q^{-k+v} \sin \frac{v+1}{2} \sin \frac{v}{2} \right| \]

\[ \leq \left( \frac{2n+1}{2\pi n} \right) \sum_{k=0}^{n} p_k q_k \left( \frac{1}{1+q} \right)^k \left( \sum_{v=0}^{k} a(v^n) q^{-k+v} \sin \frac{v+1}{2} \sin \frac{v}{2} \right) \]

\[ = O(n). \]

**Lemma-2.** \( |K_n(t)| = O \left( \frac{1}{t^2} \right), \ \text{for} \ \frac{1}{n+1} < t \leq \pi. \)

**Proof.** For \( \frac{1}{n+1} < t \leq \pi \) using Jordan’s lemma, \( \sin \frac{t}{2} \geq \frac{t}{2} \) and \( |\sin nt| \leq 1 \) we have

\[ |K_n(t)| = \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{(1+q)^n} \sum_{v=0}^{k} a(v^n) \right) q^{-k+v} \sin \left( \frac{v+1}{2} \right) \right| \]

\[ \leq \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{1+q} \right) \sum_{v=0}^{k} a(v^n) q^{-k+v} \sin \frac{v+1}{2} \sin \frac{v}{2} \right| \]

\[ \leq \frac{1}{2\pi n} \left| \sum_{k=0}^{n} p_k q_k \left( \frac{1}{1+q} \right) \sum_{v=0}^{k} a(v^n) q^{-k+v} \sin \frac{v+1}{2} \sin \frac{v}{2} \right| \]

\[ = O \left( \frac{1}{t^2} \right). \]
\[
\sum_{k=0}^{n} \binom{n}{k} q_k \frac{1}{1+q^k} \left(\sum_{v=0}^{k} (\varepsilon_v)^k q^{k-v}\right) = \frac{1}{2n+1} \sum_{k=0}^{n} p_k q_k = O\left(\frac{1}{n}\right).
\]

**Proof of Theorem 3.** By equation (2.3) we have,

\[
T_n^{\mathcal{R}_E} - f = \frac{1}{2\pi n} \sum_{k=0}^{n} p_k q_k \int_{0}^{\pi} \frac{q(t)}{\sin \frac{t}{2}} \left(\sum_{v=0}^{k} (\varepsilon_v)^k q^{k-v} \sin \left(v + \frac{1}{2}\right) t dt\right)
\]

\[
= \int_{0}^{\pi} \varphi(t) K_n(t) dt
\]

\[
= \left[ \int_{0}^{\frac{1}{n+1}} + \int_{\frac{n}{n+1}}^{1} \right] \varphi(t) K_n(t) dt
\]

\[= I_1 + I_2.
\]

Since \( f \in \text{Lip}(\alpha) \Rightarrow \varphi(t) \in \text{Lip}(\alpha), \) so by using Hölder inequality, we have

\[
I_1 = \int_{0}^{\frac{1}{n+1}} \varphi(t) K_n(t) dt
\]

Now, using Lemma 1, we have

\[
|I_1| \leq O(n) \int_{0}^{\frac{1}{n+1}} |\varphi(t)| |K_n(t)| dt
\]

\[= O(n) \int_{0}^{\frac{1}{n+1}} \frac{1}{t^\alpha} dt
\]

\[= O(n) \left[ \frac{1}{\alpha+1} \right]_{0}^{\frac{1}{n+1}}
\]

\[= O\left(\frac{1}{(n+1)^{\alpha+1}}\right)
\]

Next, using Lemma 2, we have

\[
I_2 = \int_{\frac{n}{n+1}}^{1} \varphi(t) K_n(t) dt
\]

\[
|I_2| \leq \int_{\frac{n}{n+1}}^{1} |\varphi(t)| |K_n(t)| dt
\]

\[\leq \int_{\frac{n}{n+1}}^{1} |t^\alpha| O\left(\frac{1}{n}\right) dt
\]

\[= \int_{\frac{n}{n+1}}^{1} t^{\alpha-1} dt
\]

\[= \left[ \frac{t^\alpha}{\alpha} \right]_{\frac{n}{n+1}}^{1}
\]

\[= O\left(\frac{1}{(n+1)^{\alpha}}\right)
\]

By using (4.2) and (4.3), we have

\[
|T_n^{\mathcal{R}_E} - f| = O\left(\frac{1}{(n+1)^{\alpha}}\right)
\]

Thus, if \( f \in \text{Lip}(\alpha) \) and \( 0 < \alpha < 1, \) then

\[
\|T_n^{\mathcal{R}_E} - f\| = \sup_{-\pi \leq t \leq \pi} |T_n^{\mathcal{R}_E} - f| = O\left(\frac{1}{(n+1)^{\alpha}}\right)
\]

Which completes the proof of the theorem.

**REFERENCES**


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