Abstract: Fixed point theory has many useful applications in applied sciences. The object of this paper is to obtain fixed point for continuous self mappings in Hilbert space with rational conditions.

Keywords: Hilbert space, fixed point, continuous self mapping.

AMS Classification: 47H10.

First we see some fundamental results:

Theorem 1 (a): Kannan [2] proved that “If f is self mapping of a complete metric space X into itself satisfying:

\[ d(x,y) \leq \alpha [ d(Tx, x) + d(Ty, y)], \]

for all \( x, y \in X \) and \( \alpha \in [0,1/2) \), then f has a unique fixed point in X”.

Theorem 1 (b): Fisher in [1] proved the result with:

\[ d(x,y) \leq \alpha [ d(Tx, x) + d(Ty, y)] + \beta d(x,y) \]

for all \( x, y \in X \) and \( \alpha, \beta \in [0,1/2) \), then f has a unique fixed point in X.

Theorem 1(c): A similar conclusion was also obtained by Chaterjee [3]-

\[ d(x,y) \leq \alpha [ d(Tx, x) + d(Ty, y)], \]

for all \( x, y \in X \) and \( \alpha \in [0,1/2) \), then f has a unique fixed point in X.

Theorem 1(d): Koparde and Waghmode [7] proved that “If T is a self mapping on a closed subset S of a Hilbert Space H with Kannan type condition”

\[ ||T-x-Ty|| \leq \alpha [ ||T-x||^2 + ||Ty-y||^2], \]

for all \( x, y \in S \) and \( \alpha \in [0,1/2) \), then T has a unique fixed point in S.

Sharma et.al [10] have proved the common fixed point theorems in Hilbert space with following condition-

\[ ||T-x-Ty|| \leq \alpha \left( \frac{||T-Ty||^2 + ||Ty-y||^2}{||T-x|| + ||y-Ty||} \right) + \beta ||x-y|| \]

for all \( x, y \in S, x \neq y, \alpha \in (0,1/2), \beta > 0 \) and \( 2\alpha + \beta < 1 \).

Modi G. and Gupta R.N. [12] have proved fixed point theorems in Hilbert Space, with following condition-

\[ ||T-x-Ty|| \leq \alpha \left( \frac{||T-Ty||^2 + ||Ty-y||^2}{||T-x|| + ||y-Ty||} \right) + \beta \left( \frac{||T-Ty||^2 + ||Ty-y||^2}{||T-x|| + ||y-Ty||} \right) + \gamma ||x-y|| \]

for all \( x, y \in S, x \neq y, \gamma \leq \alpha \leq 1, \beta < \frac{1}{2}, 0 \leq \gamma \) and \( 2\alpha + 2\beta + \gamma < 1 \).

Main Results

Theorem 2(a): Let S be a non empty closed subset of a Hilbert Space H. Let T be self mapping on S satisfying the condition:

\[ ||Tx-Ty|| \leq \alpha \]

\[ a \left( \frac{||T-Ty||^2 + ||Ty-y||^2}{||T-x|| + ||y-Ty||} \right) + \beta \left( \frac{||T-Ty||^2 + ||Ty-y||^2}{||T-x|| + ||y-Ty||} \right) + \gamma ||x-y|| \]

\[ \text{for all } x, y \in S, a_1, a_2, a_3, a_4 \geq 0 \text{ and } 4a_1 + 2a_2 + 2a_3 + a_4 < 1, \]

then T has a unique fixed point in S.

Proof: Let S be a non empty closed subset of a Hilbert Space H. Let T be a self mapping on S. Let \( x_0 \), S be an arbitrary point in S. We define a sequence \( \{x_n\}_{n=1}^\infty \) in S by

\[ x_{n+1} = Tx_n = T^{n+1}x_0, \]

for \( n = 0, 1, 2 \ldots \) Suppose that \( x_{n+1} \neq x_n \), for \( n = 0, 1, 2 \ldots \)

For any integer \( n \geq 1 \)

\[ ||x_{n+1} - x_n|| = ||Tx_n - Tx_n|| \]

\[ \leq a_1 \left( \frac{||x_n - x_{n-1}||^2 + ||x_{n-1} - x_{n-2}||^2 + ||x_{n-1} - Tx_n||^2}{||x_n - x_{n-1}|| + ||x_{n-1} - x_{n-2}|| + ||x_{n-1} - Tx_n||} \right) + a_2 \left( \frac{||x_n - Tx_n||^2 + ||x_{n-1} - Tx_n||^2}{||x_n - x_{n-1}|| + ||x_{n-1} - x_{n-2}|| + ||x_{n-1} - Tx_n||} \right) \]

\[ + a_3 \left( \frac{||x_n - x_{n-1}||^2 + ||x_{n-1} - x_{n-2}||^2}{||x_n - x_{n-1}|| + ||x_{n-1} - x_{n-2}||} \right) + a_4 ||x_n - x_{n-1}|| \]

\[ \leq a_1 \left( \frac{||x_n - x_{n+1}||^2 + ||x_{n-1} - x_{n}||^2 + ||x_{n-1} - x_{n+1}||^2}{||x_n - x_{n+1}|| + ||x_{n-1} - x_{n}|| + ||x_{n-1} - x_{n+1}||} \right) + a_2 \left( \frac{||x_n - x_{n+1}||^2 + ||x_{n-1} - x_{n}||^2}{||x_n - x_{n+1}|| + ||x_{n-1} - x_{n}||} \right) \]

\[ + a_3 \left( \frac{||x_n - x_{n+1}||^2 + ||x_{n-1} - x_{n}||^2}{||x_n - x_{n+1}|| + ||x_{n-1} - x_{n}||} \right) + a_4 ||x_n - x_{n-1}|| \]

\[ \Rightarrow ||x_{n+1} - x_n|| \leq k ||x_n - x_{n-1}|| \]

where \( k = \frac{2a_1 + 2a_2 + a_3 + a_4}{1 - 2a_1 - a_2 - a_3} \) and \( 4a_1 + 2a_2 + 2a_3 + a_4 < 1 \)

\[ ||x_{n+1} - x_n|| \leq k ||x_n - x_{n-1}|| \]
Let $T$ be a self mapping on $S$. Let $x_0, S$ be any arbitrary point in $S$. We define a sequence $\{x_n\}_{n=1}^\infty$ in $S$ by

$$x_{n+1} = T_n x_n \quad \text{and} \quad x_{n+2} = T_2 x_{n+1} \quad \text{for n = 0, 1, 2,} \ldots$$

Suppose that for some $n$, $x_{n+2} \neq x_n \neq x_n$, for $n = 0, 1, 2, \ldots$

For any integer $n \geq 1$

$$||x_{n+1} - x_n|| \leq \left(2a_1 + a_2 + a_3\right) \left(||x - T_n|| + ||x - T_{n-1}|| + ||x - T_{n-2}|| + ||x - T_{n-3}|| + \cdots + ||x - T_1|| + ||x - x_0||\right)$$

$$\Rightarrow \quad ||x_{n+2} - x_n|| \leq \left(2a_1 + a_2 + a_3\right) \left(||x - T_n|| + ||x - T_{n-1}|| + ||x - T_{n-2}|| + ||x - T_{n-3}|| + \cdots + ||x - T_1|| + ||x - x_0||\right)$$

Hence $v$ is a unique fixed point in $S$.
Consider

\[ \left\| v - T_1v \right\| = \left\| v - x_{2n+2} + x_{2n+2} - T_1v \right\| \]

\[ \leq \left\| v - x_{2n+2} \right\| + \left\| x_{2n+2} - T_1v \right\| \]

as \( n \to \infty \), \( x_{n+2} \to v \)

\[ \leq \left\| x_{2n+2} - T_1v \right\| \]

\[ \leq \left\| T_2x_{2n+1} - T_1v \right\| = \left\| T_1v - T_2x_{2n+1} \right\| \]

\[ \leq a_1 \left\{ \left\| v - T_1v \right\|^2 + \left\| x_{2n+1} - T_2x_{2n+1} \right\|^2 + \left\| v - T_2x_{2n+1} \right\|^2 + \left\| x_{2n+1} - T_1v \right\|^2 \right\} \]

\[ + a_2 \left\{ \left\| v - T_1v \right\|^2 + \left\| x_{2n+1} - T_2x_{2n+1} \right\|^2 \right\} \]

\[ + a_3 \left\{ \left\| v - T_2x_{2n+1} \right\|^2 + \left\| x_{2n+1} - T_1v \right\|^2 \right\} \]

\[ \leq a_1 \left\{ \left\| v - T_1v \right\|^2 + \left\| x_{2n+1} - x_{2n+2} \right\|^2 + \left\| v - x_{2n+2} \right\|^2 + \left\| x_{2n+1} - T_1v \right\|^2 \right\} \]

\[ + a_2 \left\{ \left\| v - T_1v \right\|^2 + \left\| x_{2n+1} - x_{2n+2} \right\|^2 \right\} \]

\[ + a_3 \left\{ \left\| v - x_{2n+2} \right\|^2 + \left\| x_{2n+1} - T_1v \right\|^2 \right\} \]

Hence \( v \) is a unique common fixed point of \( T_1 \) and \( T_2 \)

**CONCLUSION**

In this paper we obtained a unique fixed point for a continuous self mapping \( T \) as well as a unique common fixed point for two self mappings \( T_1 \) and \( T_2 \) satisfying rational conditions in Hilbert space.
Remark : We can get Modi G. and Gupta R.N.[12] by letting $a_1 = 0$ and Sharma, A.K., Badshah, V.H and Gupta, V.K. [10] by setting $a_1=a_3=0.$

REFERENCES