Bicomplex Version of Cayley Hamilton Theorem

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Abstract: **Theory of matrices is an integral part of algebra as well as Theory of equations. Matrices plays an important role in every branch of Physics, Computer Graphics and are also used in representing the real world's data and there are so many applications of matrices ,for this reason we thought of studying bicomplex matrices. The monograph by Price [4] contains few exercises pertaining to matrices with bicomplex entries. In this paper we discussed the bicomplex version of CAYLEY – HAMILTON THEOREM and also used it to evaluate the inverse of a non singular bicomplex matrix.**

Keywords: **Bicomplex matrices, Bicomplex polynomial, Cayley-Hamilton Theorem.**

Symbols: C_0**: set of real numbers ,**C_1**: set of complex numbers ,** C_2**: set of Bicomplex numbers.**

I. INTRODUCTION

Definition of Bicomplex Matrix:

Let A = $[\xi_{mn}]_{m\times n}$ be a bicomplex matrix, that is a matrix having bicomplex number entries.

$$A=\begin{bmatrix}\xi_{11} & \cdots & \xi_{1n}\\ \vdots & \ddots & \vdots \\ \xi_{1m} & \cdots & \xi_{mn}\end{bmatrix} \quad \xi_{pq} \in C_2,\ 1\leq p\leq m \ \&\ 1\leq q\leq n.$$

$$A=\begin{bmatrix} z_{11}+i_2w_{11} & z_{12}+i_2w_{12} & \ldots & \ldots & z_{1n}+i_2w_{1n}\\ z_{21}+i_2w_{21} & z_{22}+i_2w_{22} & \ldots & \ldots & z_{2n}+i_2w_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ z_{m1}+i_2w_{21} & z_{m2}+i_2w_{m2} & \ldots & \ldots & z_{mn}+i_2w_{mn}\end{bmatrix}$$

$$A=\begin{bmatrix} z_{11} & z_{12} & \ldots & \ldots & z_{1n}\\ z_{21} & z_{22} & \ldots & \ldots & z_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ z_{m1} & z_{m2} & \ldots & \ldots & z_{mn}\end{bmatrix} + i_2 \begin{bmatrix} w_{11} & w_{12} & \ldots & \ldots & w_{1n}\\ w_{21} & w_{22} & \ldots & \ldots & w_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ w_{m1} & w_{m2} & \ldots & \ldots & w_{mn}\end{bmatrix}$$

Where z_{mn} & w_{mn} ϵ C_1

$$A=\begin{bmatrix} x_{11} & x_{12} & \ldots & \ldots & x_{1n}\\ x_{21} & x_{22} & \ldots & \ldots & x_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ x_{m1} & x_{m2} & \ldots & \ldots & x_{mn}\end{bmatrix} + i_1 \begin{bmatrix} y_{11} & y_{12} & \ldots & \ldots & y_{1n}\\ y_{21} & y_{22} & \ldots & \ldots & y_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ y_{m1} & y_{m2} & \ldots & \ldots & y_{mn}\end{bmatrix} + i_2 \begin{bmatrix} u_{11} & u_{12} & \ldots & \ldots & u_{1n}\\ u_{21} & u_{22} & \ldots & \ldots & u_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ u_{m1} & u_{m2} & \ldots & \ldots & u_{mn}\end{bmatrix} + i_1 i_2 \begin{bmatrix} v_{11} & v_{12} & \ldots & \ldots & v_{1n}\\ v_{21} & v_{22} & \ldots & \ldots & v_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ v_{m1} & v_{m2} & \ldots & \ldots & v_{mn}\end{bmatrix}$$

where x_{mn}, y_{mn}, u_{mn} & $v_{mn} \epsilon C_0$ and $Z_{pq}=x_{pq}+i_1y_{pq}$; $w_{pq}=u_{pq}+i_1v_{pq}$

Every bicomplex matrix A=$[\xi_{mn}]_{m\times n}$ can be expressed uniquely as :

$${}^1Ae_1+{}^2Ae_2 \text{ s. t.}$$

$${}^1A=\left[{}^1\xi_{mn}\right]$$

$${}^2A=\left[{}^2\xi_{mn}\right]$$

1.1.1 Algebraic structure of Bicomplex matrices:

Let S be the set of all square matrices of order n×n define the binary composition of addition, real scalar multiplication & multiplication as follows: If $A=\begin{bmatrix} \xi_{11} & \xi_{12} & .. & .. & \xi_{1n}\\ \xi_{21} & \xi_{22} & .. & .. & \xi_{2n}\\ .. & .. & .. & .. & ..\\ .. & .. & .. & .. & ..\\ \xi_{n1} & \xi_{n2} & .. & .. & \xi_{nn}\end{bmatrix}$ &

$$B=\begin{bmatrix} \eta_{11} & \eta_{12} & .. & .. & \eta_{1n} \\ \eta_{21} & \eta_{22} & .. & .. & \eta_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ \eta_{n1} & \eta_{n2} & .. & .. & \eta_{nn} \end{bmatrix}$$ are arbitrary member of S then

$$A+B=\begin{bmatrix} \xi_{11}+\eta_{11} & \xi_{12}+\eta_{12} & \dots & \dots & \xi_{1n}+\eta_{1n} \\ \xi_{21}+\eta_{21} & \xi_{22}+\eta_{22} & \dots & \dots & \xi_{2n}+\eta_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{n1}+\eta_{n1} & \xi_{n2}+\eta_{n2} & \dots & \dots & \xi_{nn}+\eta_{nn} \end{bmatrix},$$

$$\alpha A=\begin{bmatrix} \alpha\xi_{11} & \alpha\xi_{12} & \dots & \dots & \alpha\xi_{1n} \\ \alpha\xi_{21} & \alpha\xi_{22} & \dots & \dots & \alpha\xi_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha\xi_{n1} & \alpha\xi_{n2} & \dots & \dots & \alpha\xi_{nn} \end{bmatrix} \ \&$$

A.B=

$$\begin{bmatrix} \xi_{11}\eta_{11}+\dots+\xi_{1n}\eta_{n1} & \xi_{11}\eta_{12}+\dots+\xi_{1n}\eta_{n2} & \dots & \xi_{11}\eta_{1n}+\dots+\xi_{1n}\eta_{nn} \\ \xi_{21}\eta_{11}+\dots+\xi_{2n}\eta_{n1} & \xi_{21}\eta_{12}+\dots+\xi_{2n}\eta_{n2} & \dots & \xi_{21}\eta_{1n}+\dots+\xi_{2n}\eta_{nn} \\ \dots & \dots & \dots & \dots \\ \dots & .. & \dots & \dots \\ \xi_{n1}\eta_{11}+\dots+\xi_{nn}\eta_{n1} & \xi_{n1}\eta_{12}+\dots+\xi_{nn}\eta_{n2} & \dots & \xi_{n1}\eta_{1n}+\dots+\xi_{nn}\eta_{nn} \end{bmatrix}$$

With these binary compositions S is an algebra.

Theorem 1: Let A ==$[\xi_{ij}]_{n\times n}$ be a bicomplex square matrix . Then

$$\text{Det } A = \text{Det } [{}^1A]\, e_1 + \text{Det } [{}^2A] e_2$$

Bicomplex Singular & non singular matrix:

A square matrix is said to be non singular if $|A| \notin O_2$. If $|A| \in O_2$ then A is called singular matrix.

Determinant of A is non singular then $|{}^1A| \neq 0$ & $|{}^2A| \neq 0$ (cf . Anjali [1]).

1.1.2 Let $A = [\xi_{ij}]_{n\times n}$ be bicomplex square matrix . then

$Adj[A] = Adj[{}^1A]\, e_1 + Adj\, [{}^2A]\, e_2$, This is proved by Anjali [1].

Theorem 2 (Kumar [2]) : Let A & B be two square bicomplex matrix of same order n , such that $|A| \notin O_2$ and $|B| \notin O_2$, then their product (AB) wil be invertible , and the inverse of AB will be $B^{-1}A^{-1}$.

Theorem 3 (Kumar [2]): Let A & B be two square bicomplex matrices then determinant of their product will be equal to product of their individual determinants i.e. | AB | = |A|.|B|

1.1.3 Some Properties of Bicomplex Matrices :

1) If A is any bicomplex square matrix of order n then det(A) and determinant of transpose A are equal.
2) A & B are two bicomplex matrices of order n such that B is obtained from A by interchanging any two row\ column of A then |A| = - |B|
3) If any one of row\ column in a square bicomplex matrix has each element in $O_2 = I_1 \cup I_2$ then matrix will be singular or non invertible

* Proofs of these results are straight forward (cf. Kumar [2])

1.2.1 Bicomplex Polynomial:

The polynomial of the form $P(\xi) = \sum_{k=0}^{n} \alpha_k \xi^k$ where $\alpha_k , \xi^k \in C_2$ is called bicomplex polynomial in C_2.

Zeros of bicomplex polynomial: If $P(\xi_0) = 0$ for some ξ_0 , then we say that ξ_0 is a zero of this bicomplex polynomial $P(\xi_0)$.

1.2.2 Fundamental Theorem of bicomplex algebra:

Theorem 4:

A bicomplex polynomial of degree n with non singular leading coefficient has exactly n^2 roots counted according to their multiplicities.

Proof : Let $Pn(\xi) = \sum_{k=0}^{n} \alpha_k \xi^k$, $\alpha_n \notin O_2$ be a bicomplex polynomial of degree n, If $\alpha_k = {}^1\alpha_k\, e_1 + {}^2\alpha_k\, e_2$,k = 1,2,3……..n. The roots of the polynomial $P_n(\xi)$ will be the solution of the equation $P_n(\xi) = 0$ or equivalently of the equation $\sum_{k=0}^{n} \alpha_k \xi^k = 0$. The idempotent equation is $P_n(\xi) = {}^1P_n({}^1\xi)e_1 + {}^2P_n({}^2\xi)\, e_2$ where

$${}^1P_n(\xi) = \sum_{k=0}^{n} {}^1\alpha_k\ {}^1\xi^k \ \ \& \ \ {}^2P_n(\xi) = \sum_{k=0}^{n} {}^2\alpha_k\ {}^2\xi^k$$

Thus $\sum_{k=0}^{n} {}^1\alpha\ {}^1\xi^k\, e_1 + \sum_{k=0}^{n} {}^2\alpha\ {}^2\xi^k\, e_2 = 0$

Since e_1 and e_2 are linearily independent with respect to complex coefficients.

$\sum_{k=0}^{n} {}^1\alpha_k \; {}^1\xi^k = {}^1\alpha_n \left({}^1\xi\right)^n + {}^1\alpha_{n-1} \left({}^1\xi\right)^{n-1} + \ldots\ldots\ldots + {}^1\alpha_1 \left({}^1\xi\right) + {}^1\alpha_0 = 0$ (1)

$\sum_{k=0}^{n} {}^2\alpha_k \; {}^2\xi^k = {}^2\alpha_n \left({}^2\xi\right)^n + {}^2\alpha_{n-1} \left({}^2\xi\right)^{n-1} + \ldots\ldots + {}^2\alpha_2 \left({}^2\xi\right) + {}^2\alpha_0 = 0$ (2)

Note that $\alpha_n \notin O_2 \Rightarrow {}^1\alpha_n \neq 0$ & ${}^2\alpha_n \neq 0$. Hence polynomial in (1) & (2) are of degree n .

By Fundamental Theorem of complex algebra they have precisely n roots each .

Let roots of equation (2.1) be a_1 , a_2 , ……….+$a_n \in C_1$

& roots of equation (2.2) be b_1 , b_2 , ………$b_n \in C_1$

It can be verified that bicomplex number η will be a root of the Bicomplex polynomial $P_n(\xi)$ if & only if

$\eta = a_p e_1 + b_q e_2$, $1 \le p \le n$, $1 \le q \le n$.

Hence the theorem.

II. CHARACTERISTIC VALUE PROBLEM

Given a square bicomplex matrix A of order *n* ,the problem is how to determine the scalar λ and non zero vector X which simultaneously satisfy the equation.

$$AX = \lambda X \qquad \ldots (3)$$

This is known as characteristic value problem.

Let $A=[\xi_{nn}]_{n\times n}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$

Since $\lambda X = \lambda I X$, equation (3) can be written as

$$\begin{bmatrix} \xi_{11} & \ldots & \ldots & \xi_{1n} \\ \xi_{21} & \ldots & \ldots & \xi_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \xi_{n1} & \ldots & \ldots & \xi_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda & 0 & .. & 0 \\ 0 & \lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

or $$\begin{bmatrix} \xi_{11}-\lambda & \ldots & \ldots & \xi_{1n} \\ \xi_{21} & \xi_{22}-\lambda & \ldots & \xi_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \xi_{n1} & \ldots & \ldots & \xi_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

or $(A - \lambda I)\, X = 0$ ...(4)

This is homogenous system of linear equations whose coefficient matrix is $(A - \lambda I)$. Since a non zero vector X is required, it is necessary for this coefficient matrix to have determinant equal to zero i,e $|A-\lambda I|=0$

3.1.2. Definitions:

Characteristic Equations:

The equation $|A - \lambda I| = 0$ i.e. $|{}^1A - {}^1\lambda|\, e_1 + |{}^2A - {}^2\lambda I|\, e_2 = 0$

is called characteristic equation of A.

Characteristic matrix:

The matrix $(A - \lambda I) = ({}^1A - {}^1\lambda I)\, e_1 + ({}^2A - {}^2\lambda I)\, e_2$

is called characteristic matrix of A.

Characteristic Polynomial:

The expansion of determinant $|A - \lambda I|$ yields a polynomial in λ, P(λ) which is called characteristic polynomial of matrix A.

III. CAYLEY HAMILTON THEOREM

Theorem 5: Every square bicomplex matrix A satisfies its own characteristic equation.

i.e. P(A) = 0

Proof: Let $A = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}$ where $\xi_{11}\ \xi_{12}, \xi_{21}, \xi_{22} \in C_2$

In terms of Idempotent components matrices we have

$A = {}^1A e_1 + {}^2A e_2$

i.e. $\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} {}^1\xi_{11} & {}^1\xi_{12} \\ {}^1\xi_{21} & {}^1\xi_{22} \end{bmatrix} e_1 + \begin{bmatrix} {}^2\xi_{11} & {}^2\xi_{12} \\ {}^2\xi_{21} & {}^2\xi_{22} \end{bmatrix} e_2$

Characteristic matrix of A

$$\begin{bmatrix}\xi_{11}-\lambda & \xi_{12}\\ \xi_{21} & \xi_{22}-\lambda\end{bmatrix} = \begin{bmatrix}{}^1\xi_{11}-\lambda & {}^1\xi_{12}\\ {}^1\xi_{21} & {}^1\xi_{22}-\lambda\end{bmatrix} e_1 + \begin{bmatrix}{}^2\xi_{11}-\lambda & {}^2\xi_{12}\\ {}^2\xi_{21} & {}^2\xi_{22}-\lambda\end{bmatrix} e_2$$

then $|A - \lambda I| = 0$

gives $|{}^1A - \alpha I|e_1 + |{}^2A - \beta I|e_2 = 0$

$\because$ e_1 & e_2 are linearly independent

$\Rightarrow$ $|{}^1A - \alpha I| = 0$ & $|{}^2A - \beta I| = 0$

And expansion of both sides of $|{}^1A - \alpha I| = 0$ gives polynomial $P(\alpha)$. Similarly expansion of both sides of $|{}^2A - \beta I| = 0$ gives polynomial $P(\beta)$ and since ${}^1A$ & ${}^2A$ are complex matrices so by known result we know that both these matrices satisfy their own characteristic equations.

So expansion of both sides of $|{}^1A - \alpha I| = 0$ gives

$$P(\alpha) = \alpha^2 - \alpha({}^1\xi_{11} + {}^1\xi_{22}) + {}^1\xi_{11}\,{}^1\xi_{12} - {}^1\xi_{12}\,{}^1\xi_{21} = 0$$

and expansion of both sides of $|{}^2A - \beta I| = 0$ gives

$$P(\beta) = \beta^2 - \beta({}^2\xi_{11} + {}^2\xi_{22}) + {}^2\xi_{11}\,{}^2\xi_{12} - {}^2\xi_{12}\,{}^2\xi_{21} = 0$$

and $\because$ these are complex polynomial which satisfies their own characteristic equation so

$$P({}^1A) = {}^1A^2 - {}^1A({}^1\xi_{11} + {}^1\xi_{22}) + ({}^1\xi_{11}\,{}^1\xi_{22} - {}^1\xi_{12}\,{}^1\xi_{21})I = 0$$

and

$$P({}^2A) = {}^2A^2 - {}^2A({}^2\xi_{11} + {}^2\xi_{22}) + ({}^2\xi_{11}\,{}^2\xi_{22} - {}^2\xi_{12}\,{}^2\xi_{21})I = 0$$

and $\because$ these are idempotent complex polynomial of bicomplex polynomial $P(A)$, where

$P(A) = P({}^1A)e_1 + P({}^2A)e_2$

So $P(A) = A^2 - A(\xi_{11} + \xi_{22}) + (\xi_{11}\xi_{22} - \xi_{12}\xi_{21})I = 0$

..(5)

Hence A also satisfies its own characteristics equation.

IV. IMPORTANT RESULTS

Cayley Hamilton theorem can also be used to evaluate the inverse of a non singular bicomplex mtarix and their inverse is same as the inverse obtained by adjoint method.

From(5) we have

$$(\xi_{11}\xi_{22} - \xi_{12}\xi_{21})I = A(\xi_{11} + \xi_{22}) - A^2$$

Or $I = \dfrac{1}{(\xi_{11}\xi_{22} - \xi_{12}\xi_{21})}[A(\xi_{11} + \xi_{22}) - A^2]$ (

$\because$ A is nonsingular matrix)

By pre-multiplying by A^{-1}, after simplifying we have

$$A^{-1} = \frac{1}{(\xi_{12}\xi_{21} - \xi_{11}\xi_{22})}[A - (\xi_{11} + \xi_{22})I] \quad \ldots(6)$$

Hence we get the inverse of A.

Now we proceed to show that this is same as inverse obtained by adjoint method.

From(6) , we have

$$A^{-1} = \frac{1}{(\xi_{12}\xi_{21} - \xi_{11}\xi_{22})}\left\{\begin{bmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{bmatrix} - \begin{bmatrix}\xi_{11}+\xi_{22} & 0\\ 0 & \xi_{11}+\xi_{22}\end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{(\xi_{12}\xi_{21} - \xi_{11}\xi_{22})}\begin{bmatrix}-\xi_{22} & \xi_{12}\\ \xi_{21} & -\xi_{11}\end{bmatrix}$$

or $$A^{-1} = \frac{1}{(\xi_{11}\xi_{22} - \xi_{12}\xi_{21})}\begin{bmatrix}\xi_{22} & -\xi_{12}\\ -\xi_{21} & \xi_{11}\end{bmatrix}$$

$$A^{-1} = \frac{1}{Det(A)}[adjA]$$

This result can also be obtained using the definition of inverse of a matrix (cf. Anjali [1])

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