Stability of a Quadratic-Additive Functional Equation: A Fixed Point Approach

Rajkumar.V

Hindusthan Institute of Technology, Coimbatore, Tamil Nadu, India

Abstract: **In this paper, we investigate the stability of a functional equation**

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 2f(x)+2f(y)+2f(z)+f(-x)+f(-y)+f(-z)$$

by using the fixed point theory in the sense of L. Cadariu and V. Radu.

Key words and phrases: **Hyers-Ulam stability, fixed point method, a mixed type functional equation.**

I. INTRODUCTION

In 1940, S. M. Ulam [14] raised a question concerning the stability of homomorphisms: Given a group G_1, a metric group G_2 with the metric d(.,.), and a positive number ε, does there exist a δ> 0 such that if a mapping f : $G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all x, y ϵ G_1 then there exists a homomorphism F : $G_1 \to G_2$ with

$$d(f(x), F(x)) < \varepsilon$$

for all x ϵ G1?

When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In the next year, D. H. Hyers[6] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [12] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5], [8]-[10].

Almost all subsequent proofs, in this very active area, have used Hyers' method. Namely, the function F, which is the solution of a functional equation, is explicitly constructed, starting from the given function f, by the formulae

$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ or $F(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$. We call it a direct method. In 2003, L. Cadariu and V. Radu [2] observed that the existence of the solution F for a functional equation and the estimation of the difference with the given function f can be obtained from the fixed point theory alternative. This method is called fixed point method. In 2004, they [4] applied this method to provest ability theorems of the Cauchy functional equation:

$$f(x + y) = f(x)+ f(y) \quad (1.1)$$

In 2003, they [3] obtained the stability of the quadratic functional equation:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.2)$$

by using the fixed point method. Notice that if we consider the functionsf_1; $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = ax$ and $f_2(x) = ax^2$, where a is a real constant, then f_1 satisfies the equation (1.1) and f_2 holds (1.2), respectively. We call a solution of (1.1) an additive mapping and a function satisfying (1.2) is called a quadratic map. Now we consider the functional equation:

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 2f(x)+2f(y)+2f(z)+f(-x)+f(-y)+f(-z) \quad (1.3)$$

which is called a mixed type functional equation. The function f :$\mathbb{R}\to\mathbb{R}$ defined by $f(x) = ax^2 + bx$ satisfies this functional equation, where a, b are real constants. We call a solution of (1.3) a quadratic-additive function. In 2002, S.-M. Jung [7] obtained a stability of a quadratic-additive functional equation by handling the odd part and the even part of the given function f, respectively. In his processing, he needed to take an additive mapping A which is close to the odd part $\frac{f(x)-f(-x)}{2}$ of f and a quadratic mapping Q which is approximate to the even part $\frac{f(x)+f(-x)}{2}$ of it, andthen combining A and Q to prove the existence of a quadratic-additivefunction F which is close to the given function f.

In this paper, we will prove the Hyers-Ulam stability of the quadratic-additive functional equation (1.3) by using the fixed point theory.

II. MAIN RESULTS

We recall the following result of the fixed point theory by Margolis and Diaz.

Theorem 2.1. ([11] or [13]) Suppose that a complete generalized metric space (X , d), which means that the metric d may assume infinite values, and a strictly contractive mapping J : X → X with the Lipschitz constant 0 < L < 1 are given. Then, for each given element x ϵX, either

$$d(J^n x , J^{n+1}x) = +\infty, \quad \forall n \in N \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) $d(J^n x; J^{n+1}x) < +\infty$ for all $n \geq k$;

(2) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;

(3) y^* is the unique fixed point of Y= $\{y \in x, d(J^k x, y) < +\infty$;

(4) $d(y, y^*) \leq (\frac{1}{1-L})d(y, Jy)$ for all y ϵ Y.

Throughout this paper, let X be a (real or complex) linear space and Y a Banach space. For a given mapping f : V → Y , we use the following abbreviation

Df(x, y, z) := f(x+y+z)+f(x-y)+f(y-z)+f(z-x)-2f(x)-2f(y)-2f(z)-f(-x)-f(-y)-f(-z)

for all x, y, z ϵ X. If f is a solution of the functional equation Df≡ 0,see (1.3), we call it a quadratic-additive mapping.

Stability of (1.3)

In the following theorem, we can prove the stability of the functional equation (1.3) using the fixed point theory.

Theorem 2.2.

If f : X → Y is a function such that

$$\|Df(x,y,z)\| \leq \varepsilon \qquad (3.1)$$

for all x, y, z ϵ X with a constant ε> 0 and f(0) = 0, then there exists a unique quadratic-additive mapping F : V → Y such that

$$\|f(x) - F(x)\| \leq \varepsilon \qquad (3.2)$$

for all x ϵ X. In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{f(2^n)+f(-2^n x)}{2.4^n} + \frac{f(2^n x)-f(-2^n x)}{2^{n+1}}\right) \qquad (3.3)$$

for all x ϵ X.

Proof

Let S be the set of all functions g : X → Y with g(0) = 0. We introduce a generalized metric on S by

$d(g, h) = \inf\{K\epsilon\mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varepsilon \; for\; all\; x\epsilon X\}$.

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping J : S → S, which is defined by

$$JG(X) := \frac{g(2x)-g(-2x)}{4} + \frac{g(2x)+g(-2x)}{8}$$

for all x ϵ X. Notice that

$$J^n g(x) = \frac{g(2^n x)-g(-2^n x)}{2^{n+1}} + \frac{g(2^n x)+g(-2^n x)}{2.4^n}$$

for all n ϵ N and x ϵ X. Let g, h ϵ S and let K ϵ [0,∞] be an arbitraryconstant with d(g, h) ≤ K. From the definition of d, we have

$$\|Jg(x) - Jh(x)\| \leq \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \leq \frac{K}{2}\varepsilon$$

for all x ϵ X, which implies that

$d(Jg,Jh) \leq \frac{1}{2} d(g,h)$

for any g, h ϵ S. That is, J is a strictly contractive self-mapping of Swith the Lipschitz constant $\frac{1}{2}$. Moreover, by (3.1) we see that

$$\|f(x) - Jf(x)\| = \frac{1}{8}\|3Df(x,x,-x) - Df(-x,-x,x)\| \leq \frac{\varepsilon}{2}$$

for all x ϵ X. It means that $d(f,Jf) \leq \frac{1}{2} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges tothe unique fixed point F : X → Y of J in the set T = $\{g\epsilon S \setminus d(f,g) < \infty\}$, which is represented by (3.3) for all x ϵ X. Notice that

$$d(f,F) \leq \frac{1}{1-\frac{1}{2}}\, d(f,Jf) \leq \varepsilon$$

Which implies (3.2). By the definition of F, together with (3.1) and (3.3), we have

$$\|DF(x,y,z)\| = \lim_{n\to\infty}\left\|\frac{Df(2^n x,2^n y,2^n z) - Df(-2^n x,-2^n y,-2^n z)}{2^{n+1}} + \frac{Df(2^n x,2^n y,2^n z) + Df(-2^n x,-2^n y,-2^n z)}{2.4^n}\right\|$$

$$= \lim_{n\to\infty}\frac{(2^n+1)\varepsilon}{2.4^n} = 0$$

for all x, y, z ϵ X. For a given mapping f : X → Y , we use the following abbreviations

Af(x, y) := f(x + y) - f(x) - f(y),

Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)

for all x, y ϵ X. Using Theorem 3.1, we will show the stability results of the additive functional equation Af ≡ 0 and the quadratic functionalequation Qf ≡ 0 in the following corollaries.

Corollary 3.2

If f : X → Y is a function such that

$$\|Af(x,y)\| \leq \varepsilon \qquad (3.4)$$

for all x, y ϵ X with a constant ε> 0 and f(0) = 0, then there exists a unique additive mapping F : X → Y such that

$$\|f(x) - F(x)\| \leq 7\varepsilon \qquad (3.5)$$

for all x ϵ X. In particular, the mappings F is represented by

$$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \qquad (3.6)$$

for all x ϵ X.

Proof.

Notice that

$$\|Df(x,y,z)\| = \|Af(x,y+z) - Af(x-y,y-z) - Af(z-x,x-z) + Af(y,z) + Af(x,-x) + Af(y,-y) + Af(z,-z)\| \le 7\varepsilon$$

for all x, y, z ϵ X. Therefore, according to Theorem 3.1, there exists a unique mapping F : X → Y satisfying (3.2), which is represented by(3.3). Observe that

$$\lim_{n\to\infty} \left\| \frac{f(2^n x) + f(-2^n x)}{2^{n+1}} \right\| = \lim_{n\to\infty} \left\| \frac{f(2^n x) + f(-2^n x) - f(0)}{2^{n+1}} \right\|$$

$$= \lim_{n\to\infty} \frac{1}{2^{n+1}} \|Af(2^n x, -2^n x)\|$$

$$\le \lim_{n\to\infty} \frac{1}{2^{n+1}} \varepsilon = 0$$

as well as

$$\lim_{n\to\infty} \left\| \frac{f(2^n x) + f(-2^n x)}{2.4^n} \right\| \le \lim_{n\to\infty} \frac{\varepsilon}{2.4^n} = 0$$

for all x ϵ X. From this and (3.3), we get (3.6). Moreover, we have

$$\left\| \frac{Af(2^n x, 2^n y)}{2^n} \right\| \le \frac{\varepsilon}{2^n}$$

for all x, y ϵ X. Taking the limit as n →∞ in the above inequality, we get

$$AF(x, y) = 0$$

for all x, y ϵ X.

Corollary 2.4.

If f : X → Y is a function such that

$$\|Qf(x,y)\| \le \varepsilon \qquad (3.7)$$

for all x, y ϵ X with a constant ε > 0 and f(0) = 0, then there exists a unique quadratic mapping F : X → Y such that

$$\|f(x) - F(x)\| \le \frac{26}{4}\varepsilon \qquad (3.8)$$

for all x ϵ X. In particular, the mappings F is represented by

$$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \qquad (3.9)$$

for all x ϵ X.

Proof.

Notice that

$$\|Df(x,y,z)\| = \frac{1}{4}\|2Qfx + z, y) - Qf(x-y,z) - Qfx-y,-z+2Qfx,y+z+4Qfz,x+4Qfy,z-4Qf0,x-4Qf0,y-4Qf(0,z) \le 264\varepsilon$$

for all x, y, z ϵ X. According to Theorem 3.1, there exists a unique mapping F : V → Y satisfying (3.8) which is represented by (3.3).

Observe that

$$\lim_{n\to\infty} \left\| \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right\| = \lim_{n\to\infty} \|Qf(0, 2^n x)\| \le \lim_{n\to\infty} \varepsilon 2n+1 = 0$$

for all x ϵ X. From this and (3.3), we get (3.9) for all x ϵ X. Moreover, we have

$$\left\| \frac{Qf(2^n x, 2^n y)}{4^n} \right\| \le \frac{\varepsilon}{4^n}$$

for all x, y ϵ X. Taking the limit as n →∞ in the above inequality, weget

$$QF(x,y) = 0$$

for all x, y ϵ X.

REFERENCES

[1]. T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math.Soc. Japan 2 (1950), 64{66.

[2]. L. C¸adariu and V. Radu, Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math.4(1) (2003), Art. 4.

[3]. L. Cadariu and V. Radu, Fixed points and the stability of quadratic functional equations, An. Univ. Timisoara Ser. Mat.-Inform. 41 (2003), 25{48. Stability of approximately quadratic-additive mappings 49

[4]. L. C¸adariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach in Iteration Theory, Grazer MathematischeBerichte, Karl-Franzens-Universit Äaet, Graz, Graz, Austria 346 (2004), 43{52.

[5]. P. G·avruta, A generalization of the Hyers-Ulam-Rassias stability of approxi-mately additive mappings, J. Math. Anal.Appl. 184 (1994), 431{436.

[6]. D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U. S. A. 27 (1941), 222{224.

[7]. S.-M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, J. Math. Anal.Appl. 222 (1998), 126{137.

[8]. H.-M. Kim, On the stability problem for a mixed type of quartic and quadratic functional equation, J. Math. Anal.Appl. 324 (2006), 358{372.

[9]. Y.-H. Lee, On the stability of the monomial functional equation, Bull. Korean Math.Soc. 45 (2008), 397{403.

[10]. Y. H. Lee and K. W. Jun, On the stability of approximately additive mappings, Proc. Amer. Math. Soc. 128 (2000), 1361{1369.

[11]. B. Margolis and J. B. Diaz, A fixed point theorem of the alternative for con-tractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305{309.

[12]. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer. Math. Soc. 72 (1978), 297{300.

[13]. I. A. Rus, Principles and Applications of Fixed Point Theory, Ed. Dacia, Cluj- Napoca 1979 (in Romanian).

[14]. S. M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1968, p. 63.